APPENDIX C

CONTINUOUS-TIME MARKOV MODEL

The differential equations Eq. (3.48) – (3.50) in Chapter 3 can be solved using Laplace transforms as follows. However to avoid confusion with the notation, s that usually used in Laplace transform, the variable S is changed to X

$$\frac{dX}{dt} = -\phi X + \omega F \tag{C1}$$

$$\frac{dF}{dt} = \phi X - (\lambda + \omega)F \tag{C2}$$

$$\frac{dD}{dt} = \lambda F \tag{C3}$$

Applying Laplace for both sides of the equations yields

$$sX - x(0) = -\phi X + \omega F$$

$$sF - f(0) = \phi X - (\lambda + \omega)F$$

$$sD - d(0) = \lambda F$$

With the initial condition $x(0) = 1,\ d(0) = 0$ and $f(0) = 0$, the equation becomes

$$sX - 1 = -\phi X + \omega F$$

$$sF = \phi X - (\lambda + \omega)F$$

$$sD = \lambda F$$

Rearrange the equations into a form suitable for solving X, F and D yields

$$(s+\phi)X = \omega F + 1$$

(c1)

$$\phi X = (s+\lambda+\omega)F$$

(c2)

$$sD = \lambda F$$

(c3)

Suppose F is to be eliminated in Eq. (c1). By substituting $F = \left(\dfrac{\varphi}{s+\lambda+\omega}\right)X$ from Eq. (c2) yields

$$X = \frac{s+\lambda+\omega}{s^2+(\phi+\lambda+\omega)s+\phi\lambda}$$

(c4)

Substituting Eq. (a4) back in Eq. (c2) to solve for F yields

$$F = \frac{\phi}{s^2+(\phi+\lambda+\omega)s+\phi\lambda}$$

(c5)

Solve for D by substituting Eq. (c5) in Eq. (c3), yields

$$D = \frac{\lambda\phi}{s\left(s^2+(\phi+\lambda+\omega)s+\phi\lambda\right)}$$

Then, the function $s^2+(\phi+\lambda+\omega)s+\phi\lambda = 0$ needs to be decomposed in order to find the roots for s. The roots for a quadratic equation, $ax^2+bx+c=0$, is in the form of

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2}\ .$$

Let $a=1$, $b=\phi+\lambda+\omega$ and $c=\phi\lambda$. Thus, the roots for s are

$$s_1 = \frac{-(\phi+\lambda+\omega)+\sqrt{(\phi+\lambda+\omega)^2-4\phi\lambda}}{2}$$

and

$$s_2 = \frac{-(\phi+\lambda+\omega) - \sqrt{(\phi+\lambda+\omega)^2 - 4\phi\lambda}}{2}$$

Let denote $s_1 = r_1$ and $s_2 = r_2$. Therefore, the equation $s^2 + (\phi+\lambda+\omega)s + \phi\lambda = (s-r_1)(s-r_2)$.

To find the function $x(t)$ that has the Laplace transform

$$X(s) = \frac{s+\lambda+\omega}{s^2+(\phi+\lambda+\omega)s+\phi\lambda} = \frac{s+\lambda+\omega}{(s-r_1)(s-r_2)},$$

the function is decomposed into a sum of fractions as follows

$$\frac{s+\lambda+\omega}{s^2+(\phi+\lambda+\omega)s+\phi\lambda} = \frac{A}{(s-r_1)} + \frac{B}{(s-r_2)}.$$

Multiplication of each term by $(s-r_1)(s-r_2)$ yields

$$s+\lambda+\omega = A(s-r_2) + B(s-r_1)$$

Substitute $s = r_1$ yields

$$r_1+\lambda+\omega = A(r_1-r_2)$$

$$A = \frac{r_1+\lambda+\omega}{(r_1-r_2)}$$

Substitute $s = r_2$ yields

$$r_2+\lambda+\omega = B(r_2-r_1)$$

$$B = \frac{r_2+\lambda+\omega}{(r_2-r_1)} = -\frac{r_2+\lambda+\omega}{(r_1-r_2)}$$

Thus, the result is that

$$X(s) = \frac{s+\lambda+\omega}{s^2+(\phi+\lambda+\omega)s+\phi\lambda} = \frac{r_1+\lambda+\omega}{(r_1-r_2)}\left(\frac{1}{s-r_1}\right) - \frac{r_2+\lambda+\omega}{(r_1-r_2)}\left(\frac{1}{s-r_2}\right)$$

having the inverse Laplace of

$$x(t) = \frac{r_1 + \lambda + \omega}{(r_1 - r_2)} e^{r_1 t} - \frac{r_2 + \lambda + \omega}{(r_1 - r_2)} e^{r_2 t}$$

$$x(t) = \frac{1}{(r_1 - r_2)} \left[(r_1 + \phi) e^{r_2 t} - (r_2 + \phi) e^{r_1 t} \right]$$

To find the function $f(t)$ having the Laplace transform

$$F(s) = \frac{\phi}{s^2 + (\phi + \lambda + \omega)s + \phi\lambda} = \frac{\phi}{(s - r_1)(s - r_2)}$$

the function is decomposed into a sum of fractions as follows:

$$\frac{\phi}{s^2 + (\phi + \lambda + \omega)s + \phi\lambda} = \frac{A}{(s - r_1)} + \frac{B}{(s - r_2)}$$

Multiplication of each term by $(s - r_1)(s - r_2)$ yields

$$\phi = A(s - r_2) + B(s - r_1)$$

Substitute $s = r_1$ yields

$$\phi = A(r_1 - r_2)$$

$$A = \frac{\phi}{(r_1 - r_2)}$$

Substitute $s = r_2$ yields

$$\phi = B(r_2 - r_1)$$

$$B = \frac{\phi}{(r_2 - r_1)} = -\frac{\phi}{(r_1 - r_2)}$$

The result is that

$$F(s) = \frac{\phi}{s^2 + (\phi + \lambda + \omega)s + \phi\lambda} = \frac{\phi}{(r_1 - r_2)} \left(\frac{1}{s - r_1} \right) - \frac{\phi}{(r_1 - r_2)} \left(\frac{1}{s - r_2} \right)$$

with the inverse Laplace of

$$f(t)=\frac{\phi}{(r_1-r_2)}e^{r_1 t}-\frac{\phi}{(r_1-r_2)}e^{r_2 t}$$

$$f(t)=\frac{\phi}{(r_1-r_2)}\left(e^{r_1 t}-e^{r_2 t}\right)$$

To find the function $d(t)$ having the Laplace transform

$$D(s)=\frac{\lambda\phi}{s\left(s^2+(\phi+\lambda+\omega)s+\phi\lambda\right)}=\frac{\lambda\phi}{s(s-r_1)(s-r_2)}$$

the function is decomposed into a sum of fractions as follows:

$$\frac{\lambda\phi}{s\left(s^2+(\phi+\lambda+\omega)s+\phi\lambda\right)}=\frac{\lambda\phi}{s(s-r_1)(s-r_2)}=\frac{A}{s}+\frac{B}{(s-r_1)}+\frac{C}{(s-r_2)}$$

Multiplication of each term by $s(s-r_1)(s-r_2)$ yields

$$\phi=A(s-r_1)(s-r_2)+Bs(s-r_2)+Cs(s-r_1)$$

Substitute $s=0$ yields

$$\lambda\phi=Ar_1r_2$$

$$A=\frac{\lambda\phi}{r_1r_2}$$

Substitute $s=r_1$ yields

$$B=\frac{\lambda\phi}{r_1(r_1-r_2)}$$

Substitute $s=r_2$ yields

$$C=-\frac{\lambda\phi}{r_2(r_1-r_2)}$$

The result is that

$$D(s)=\frac{\lambda\phi}{s(s-r_1)(s-r_2)}=\frac{\lambda\phi}{r_1r_2}\left(\frac{1}{s}\right)+\frac{\lambda\phi}{r_1(r_1-r_2)}\left(\frac{1}{s-r_1}\right)-\frac{\lambda\phi}{r_2(r_1-r_2)}\left(\frac{1}{s-r_2}\right)$$

and the inverse Laplace of $D(s)$ is

$$d(t)=\frac{\lambda\phi}{r_1 r_2}+\frac{\lambda\phi}{r_1(r_1-r_2)}e^{r_1 t}-\frac{\lambda\phi}{r_2(r_1-r_2)}e^{r_2 t}$$

$$d(t)=1-\frac{1}{(r_1-r_2)}\left(r_1 e^{r_2 t}-r_2 e^{r_1 t}\right)$$

Thus, the time dependent solutions for the state probabilities are given by

$$S(t)=\frac{1}{(r_1-r_2)}\left[(r_1+\phi)e^{r_2 t}-(r_2+\phi)e^{r_1 t}\right]$$

$$F(t)=\frac{\phi}{(r_1-r_2)}\left(e^{r_1 t}-e^{r_2 t}\right)$$

$$D(t)=1-\frac{1}{(r_1-r_2)}\left(r_1 e^{r_2 t}-r_2 e^{r_1 t}\right)$$

where the term r_1 and r_2 are defined as

$$r_1=\frac{-(\phi+\lambda+\omega)+\sqrt{(\phi+\lambda+\omega)^2-4\phi\lambda}}{2}$$

and

$$r_2=\frac{-(\phi+\lambda+\omega)-\sqrt{(\phi+\lambda+\omega)^2-4\phi\lambda}}{2}$$