Analytical Method for Free and Forced Vibration of Simply Supported Beam under Moving Load

A slender beam, made from a uniform homogeneous and isotropic material with length ℓ, flexural rigidity EI, and having a prismatic cross-section with constant mass per unit length $\overline{m}$ acted upon by a concentrated force P moving at a constant speed V is shown in **Figure C.1**. It is assumed that initially, at time $t = 0$, the force is situated at the left-end support.

The behavior of the beam is assumed to be governed by the Euler-Bernoulli beam theory, then the the governing differential equation describing the lateral vibration of a beam carrying the moving concentrated force P is [90]

$$EI\frac{\partial^4 y}{\partial x^4} + \overline{m}\frac{\partial^2 y}{\partial t^2} = \delta(\ell - x)P(t) \quad \textbf{(C.1)}$$

where y is the deflection of the beam measured upwards from its equilibrium position when unloaded, and $\delta(\cdots)$ the Dirac delta function.

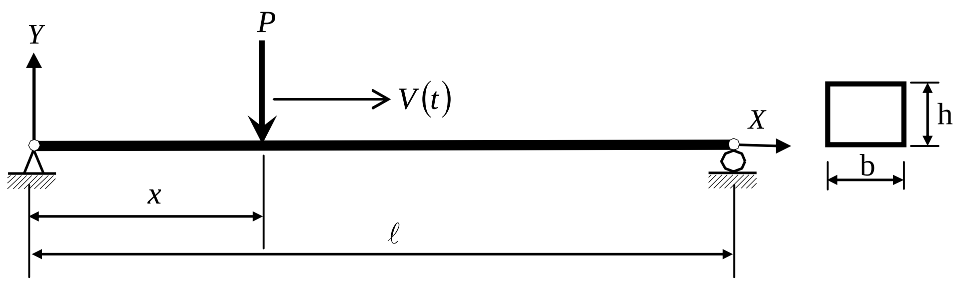


Figure C.1 Simply supported beam induced by a moving force

Before analyzing the response of the simply supported beam to moving load, it is important to study the free vibration of the beam. The free vibration analysis is here an essential first step in obtaining the forced vibration response of the beam as presented in the following subsection.

C.1 Free Vibration

The differential equation governing the free vibration of the simply supported beam is obtained by setting the right-hand side in **Equation C.1** equal to zero, giving homogen differential equation:

$$EI\frac{\partial^4 y}{\partial x^4}+\overline{m}\frac{\partial^2 y}{\partial t^2}=0 \qquad \textbf{(C.2)}$$

Solution for this type of equation is the separation of the variables which assumes that the solution is the product of two functions, one defines the deflection shape and the other defines the amplitude of vibration with time:

$$y(x,t)=\Phi(x)f(t) \qquad \textbf{(C.3)}$$

By substituting **Equation (C3)** into **Equation (C2)** yields:

$$EI\ f(t)\frac{\partial^4\Phi(x)}{\partial x^4}+\overline{m}\,\Phi(x)\frac{\partial^2 f(t)}{\partial t^2}=0 \qquad \textbf{(C.4)}$$

Rearranging **Equation (C.4)** yields:

$$\frac{EI}{\overline{m}}\frac{\Phi^{IV}(x)}{\Phi(x)}=-\frac{\ddot{f}(t)}{f(t)^{\bullet}} \qquad \textbf{(C.5)}$$

where roman index indicates derivative to x, while dot index is derivative to t. Since each of the variables x and t are independent variables, then each side of the **Equation (C.5)** is equal to a constant, say ω^2, it may be rewritten down to two ordinary differential equations that have to be satisfied:

$$\Phi^{IV}(x)-a^4\Phi(x)=0 \qquad \textbf{(C.6)}$$

$$\ddot{f}(t)+\omega^2 f(t)=0 \qquad \textbf{(C.7)}$$

where

$$a^4=\frac{\overline{m}\omega^2}{EI} \qquad \textbf{(C.8)}$$

The value of ω^2 is as follows:

$$\omega = C\sqrt{\frac{EI}{\overline{m}L^4}}, \quad C = (aL)^4 \tag{C.9}$$

Equation (C.7) is free vibration equation for undamped single degree of freedom with the general solution is given by:

$$f(t) = A\cos\omega t + B\sin\omega t \tag{C.10}$$

where A and B are integration constants. The general solution of **Equation (C.6)** is given by:

$$\Phi(x) = Ce^{sx} \tag{C.11}$$

Substituting **Equation (C.11)** into **Equation (C.6)** yields:

$$(s^4 - a^4)Ce^{sx} = 0 \tag{C.12}$$

The auxiliary equation is:

$$s^4 - a^4 = 0 \tag{C.13}$$

Analysing Equation (C.13) yields:

$$s_1 = a, \quad s_2 = -a, \quad s_3 = ia, \quad s_4 = -ia \tag{C.14}$$

Substituting **Equation (C.14)** into **Equation (C.11)** yields the general solution:

$$\Phi(x) = C_1e^{ax} + C_2e^{-ax} + C_3e^{iax} + C_4e^{-iax} \tag{C.15}$$

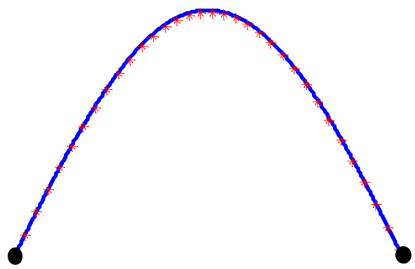
where C_1, C_2, C_3, C_4 are integration constants. Exponential functions in **Equation (C.15)** can be defined in trigonometric and hyperbolic function as follows:

$$\begin{aligned} e^{\pm ax} &= \cosh ax \pm \sinh ax \\ e^{\pm iax} &= \cos ax \pm i\sin ax \end{aligned} \tag{C.16}$$

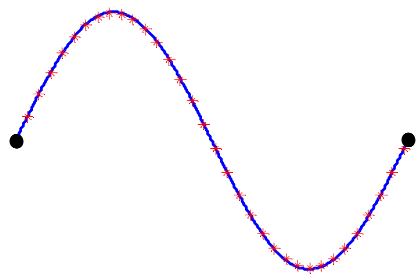
Substituting **Equation (C.16)** into **Equation (C.15)** yields:

$$\Phi(x) = A\sin ax + B\cos ax + C\sinh ax + D\cosh ax \tag{C.17}$$

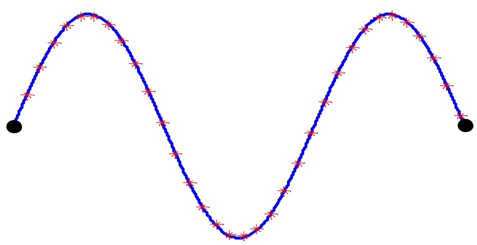
where A, B, C and D are new integration constants. The four integration constants determine the magnitude of vibration that would require acknowledging of the initial conditions of motions. The first four and mode shape of simply supported beam are shown in **Figure C.2** and **Table C.1**.



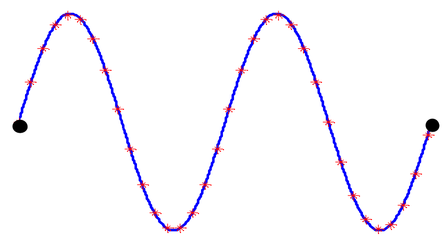
(a) Mode 1, 34.821 Hz



(b) Mode 2, 139.283 Hz



(c) Mode 3, 313.388 Hz



(d) Mode 4, 557.135Hz

Figure C.2 The first four mode shapes of simply supported beam (—) analytical; (***) present code using FEM

Table C.1 Natural frequency comparison

Mode shape	Theoritical frequency (Hz)	Present code frequency (Hz)	ANSYS frequency (Hz)
1	34.821	34.821	34.818
2	139.283	139.284	139.234
3	313.388	313.398	313.140
4	557.135	557.194	556.374

It can be seen from **Table C.1** that the natural frequencies of simply supported beam obtained from present code under Matlab® are closer to the frequencies obtained from the theoretical analysis. The results are also verified with ANSYS to the mesh size 20 elements.

C.2 Forced Vibration under Moving Load

Derivation of the equation for m otion for any point and moment in time can be shown to be, with the assumption of only light damping, $\beta = 0.1$ which is practical in most situations [90].

$$y(x,t) \approx y_0 \frac{1}{2n^4}\left[e^{-\omega_n t}\, sin\, n\omega t - \frac{n^2}{\beta} cos\, n\omega t\left(1 - e^{-\omega_n t}\right)\right] sin\frac{n\pi x}{\ell} + y_0 \sum_{\substack{n=1 \\ n\neq\gamma}}^{\infty} sin\frac{n\pi x}{\ell}\frac{1}{n^2\left(n^2-\alpha^2\right)}\left(sin\, n\omega t - \frac{\alpha}{n}e^{-\omega_n t}\, sin\,\omega_n t\right) \quad \textbf{(C.18)}$$

where $\alpha = \gamma$, $\gamma = 1,2,3\cdots$ and y_0 is the deflection of the beam at the center point,

$$y_0 = \frac{P\ell^3}{48EI} \quad \textbf{(C.19)}$$

The simply supported beam is solved by using the exact analytical model, **Equation (C.18)**, the present code under Matlab, and the ANSYS softwares. From **Figure C.3,** it can be concluded that the result from the present code is in very good agreement with those obtained using the exact analytical model and the ANSYS.

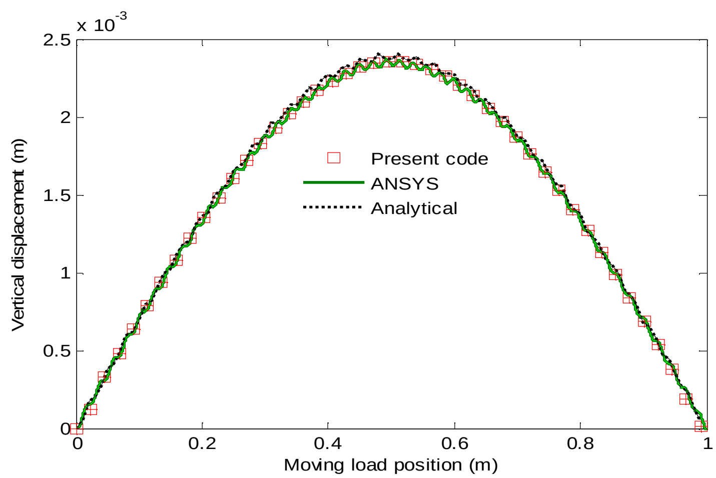


Figure C.3 Dynamic response of a simply supported beam induced by a moving force