RETRACTABLE MECHANISM OF FOLDABLE BAR STRUCTURE

By

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Dissertation submitted in partial fulfilment of

the requirements for the

Bachelor of Engineering (Hons)

(Civil Engineering)

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CERTIFICATION OF APPROVAL

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A project dissertation submitted to the
Civil Engineering Programme
Universiti Teknologi PETRONAS
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(Civil Engineering)

Approved:

Mr Muhammad Sanif Maulut
Project Supervisor

UNIVERSITI TEKNOLOGI PETRONAS
TRONOH, PERAK

CERTIFICATION OF ORIGINALITY

This is to certify that I am responsible for the work submitted in this project, that the original work is my own except as specified in the references and acknowledgements, and that the original work contained herein have not been undertaken or done by unspecified sources or persons.

Emma Noor Aireen binti Abdul Rahim

ABSTRACT

Studying and evaluating retractable mechanism of foldable bar structure is the main focus in this project. This study has been narrowed down to studying theory of foldable bar structure using scissor hinge system. Scissor hinge system is a system which consists of rigid, flat plates, connected by joints, allows relative rotation about one axis. Perhaps, this system could solve the mystery towards convertible structures in Malaysia to suit Malaysia's wet and humid weather. Of course, engineering behaviours and other alternative application/structure would be the main tasks for this project and that will be explained further later in this paper. The main scope of study is study the system and evaluate. The methods of this study are firstly, to collect and read as many reading materials related to retractable structures as possible. Some methods have been modified to suit the progress of this project. This can be found later in methodology chapter. Afterwards, finding the alternative application to use the retractable mechanism onto other structure would be investigated. However, the scope of study may or may not extend to other deliverables, depending on time of progress. On chapter two (2) would tell the basic findings and general concept of the study. Some detail information on Hoberman's Angulated Element also being stated and discussed in this paper.

ACKNOWLEDGEMENTS

First of all, I would like to thank my supervisor, Mr Muhammad Sanif Maulut, for his advice, encouragement and support. He introduced me to the subject of retractable structures. The regular discussion with him has been very beneficial to my research.

I would also like to forward my thankfulness to Dr Victor Macam and Dr Mazlan Napiah of Universiti Teknologi PETRONAS for commenting and suggesting what I should do for this study.

Appreciation also goes to my colleagues; Amier Avril Leiking and Mohd Zakuan Md Nor that have been a helping hand when needed during the design period using software and also mere fabrication for experiment.

I'm also grateful to both of my parents with endless support and gave me much motivation to work hard. The financial aid in order to pursue this project was also given by my parents.

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LIST OF ABBREVIATIONS

GAE	Generalized Angulated Element
FYP	Final Year Project
USA	United States of America

CHAPTER 1
INTRODUCTION

Final Year Project II focuses on the continuation of the tasks done from Final year Project I which had been done last semester. Since some of the main objectives for this project have been accomplished, new sets of objectives have been provided in order to narrow down comprehension. For Final Year Project II, basically is to find alternatives or other type of structures with the purpose of applying known and studied scissor hinge mechanism, mainly Hoberman's method, onto it.

1.1 Background of Study

This study is about evaluating the structural integrity and structural analysis on a type of foldable bar structures which has not been practiced yet in Malaysia. However, in foreign countries like USA for example, they have been using this foldable bar technique for years now and already have several buildings with roof using the same concept of foldable bars. Deployable structures are a novel and unique type of engineering structure, whose geometry can be altered to meet practical requirements. Large aerospace structures, e.g. antennas and masts, are prime examples of deployable structures.

The purpose of this study is to propose and introduce our nation to the retractable mechanism of foldable bars, which in fact is the latest trend and somehow economically convenient in the long run. The extension of this project is to find alternatives structure, other than the normal retracting roof to be proposed so that the retracting mechanism can be applied onto it.

1.2 Problem Statement

1.2.1 Problem Identification

Malaysia is a developing country and day by day, the land space is getting smaller. Hence, building up separate buildings for different occasions seems like wasting valuable space. For instance, a stadium. Why do we need separate covered and uncovered stadiums for different occasions and sports while we can improvise it to be only one?

To add up the current problem of land space, the factor of Malaysia's weather also should be taken into consideration. As we all know, Malaysia is well known as wet and humid country. The rain may come and go at any time and sometimes without any warning and signs. Thus Malaysians need some shed for protection.

It is sometimes inconvenient to manually configure a structure whenever it is needed to be retracted or extended to suit different occasions and environment conditions. Furthermore, the aesthetic value of structure and architectural has not been improved recently. All buildings have the same visual quality if to be compared to foreign countries' structures.

1.2.2 Significant of Project

The main goal of this project is to study in detail all mechanisms involve in retractable and foldable bar structures. Furthermore, with the concept learned later on, the project will then extend to apply known concept onto other structure that might come in handy when it is retractable. This study also involves proving the theory is absolutely applicable to structures so that the structure can be retractable and extendable.

1.3 Objective and Scope of Study

1.3.1 Objectives

The main objective is to study and evaluate the mechanism and principle behind retractable structure or foldable structure using specific system called scissor hinge system whereby the structure consisting of rigid, flat plates, connected by joints that allow only relative rotation about one axis.

This study also consists of exploring its engineering behaviour and develops some alternatives/other types of structures for the theory to be applied to. Afterwards, the project is preceded to constructing roof structures using the same principles that has been studied and evaluated. The project also could be extended to constructing roof structures using the same principles that has been studied and evaluated if time is adequate to implement it.

1.3.2 Scope of Study

The extent of this study would be to evaluate and analyze structures whereby this theory of scissor hinge would be used. This study would include on how the structure would behave if the theory is being applied on it. Also, the study would find a solution or modification to be made on the theory so that it would be applicable to other structures. The study would also be focused on finding the right type, size, length and shape of the bars. At the end of these findings, this project would also include proposal of applying theory onto other engineering structures.

CHAPTER 2
LITERATURE REVIEW AND THEORY

2.1 Literature Review

In the 1960s and '70s, many cities built doughnut-shaped multipurpose stadiums. It was also the era of the domed stadium. But today, the latest trend for stadiums is the retractable roof stadiums. This latest trend of convertible stadiums allows athletes and sportsman to play in the field no matter what the weather condition is occurring. Now, this convertible concept has been modified using geometric design of symmetric expandable structures consisting of rigid, flat plates connected to joints that allow only a relative rotation about one axis which is called the *scissor hinges or joints* [1].

Looking at these types of structures made it looked very interesting in assemblies of identical plates forming a complete ring, and which are able to move relative to one another between two extreme configurations. In the closed configuration these structures from a gap free disk and in the open configuration they form an annulus with a circular opening in the middle [2].

This piece of work has been similar to Florian Kovdes and TIbor Tarnat from University of Budapest but the only difference between the original findings of Frank Jensen and Sergio Pellegrino was the application on a spherical structure. Simple expandable structures based on the concept of pantographic elements, whereby straight bars connected through scissor hinges have been known for a long time. One of the simplest forms of such pantographic structures is well known lazy tong in which a series of pantographic elements are connected to form two-dimensional linearly extendible structures [3].

For now, this study would focus on Hoberman's (1990) invention of the simple angulated element consisting of a pair of identical angulated rods connected together by a scissor hinge. This element has special property that lines through the end nodes subtend to a constant angle when the angle between two angulated bars is changed. An angulated element subtends a constant angle when it is opened or closed. See figure for theory visual.

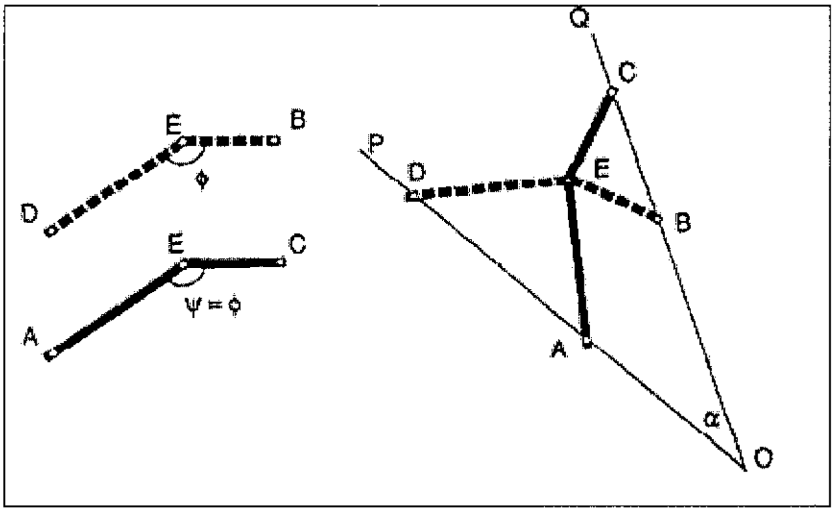


Figure 1 : Hoberman's element, identical angulated rods.

2.2 Theory

2.2.1 Hoberman's Angulated Element

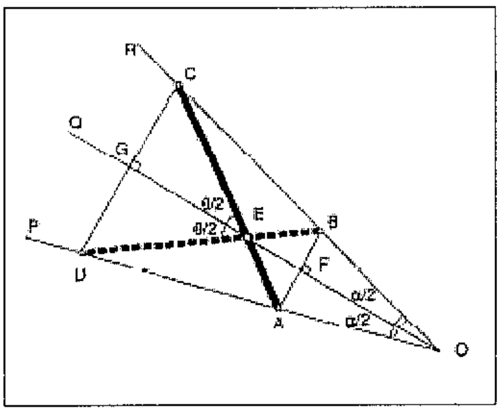


Figure 2 : Ordinary Pantographic Element, Identical Straight rods

This element is symmetric at line OQ. Angle α and angle θ can be obtained by these relationships:

$$\overline{CG} - \overline{BF} = \overline{FG} \tan \alpha/2$$

$$\overline{CG} = \overline{CE} \sin \theta/2$$

$$\overline{BF} = \overline{BE} \sin \theta/2 = \overline{AE} \sin \theta/2$$

$$\overline{FG} = \overline{AC} \cos \theta/2.$$

$$(\overline{CE} - \overline{AE}) \sin \theta/2 = \overline{AC} \cos \theta/2 \tan \alpha/2$$

$$\tan \alpha/2 = \frac{\overline{CE} - \overline{AE}}{\overline{AC}} \tan \theta/2.$$

Figure 3 : Relationships/Equations accompany Hoberman's Element

2.2.2 *Generalised Angulated Elements*

Generalised Angulated Elements (GAE) is a set of interconnected angulated roads that form a chain of any number of parallelograms with either isosceles triangle, which is Type I GAE, or similar triangles, Type II GAE, at either end. When folded or expanded, the angles would remain constant.

2.2.2.1 Type I GAE

Firstly, before explaining the theory behind Type I GAE, let's look at angulated element below:

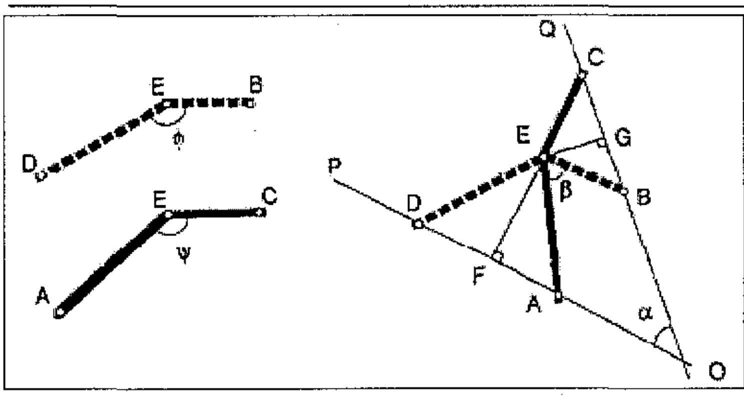


Figure 4 Simplest Type I GAE

Which accompanied by these relationships:

$$\overline{AE} = \overline{DE} \text{ , } \overline{BE} = \overline{CE}$$
$$\alpha = 180^\circ - (\angle AEF + \beta + \angle BEG)$$

Where we can assume that ψ is equal to $\varnothing$. Triangle ADE and BCE are isosceles triangles. Therefore,

$$\angle AEF = \frac{\phi - \beta}{2}$$
$$\angle BEG = \frac{\psi - \beta}{2}$$

Hence,

$$\alpha = 180^\circ - \frac{\phi + \psi}{2} = \text{constant}$$

This indeed indicates that this element subtends a constant angle. In simpler form, it means that when the element is retracted or folded, the angle will remain constant. Type I GAE is obtained when either ψ OR $\varnothing$ is equal to 180°. Figure 5 will illustrate Type I GAE consisting of two isosceles triangles connected by two parallelograms.

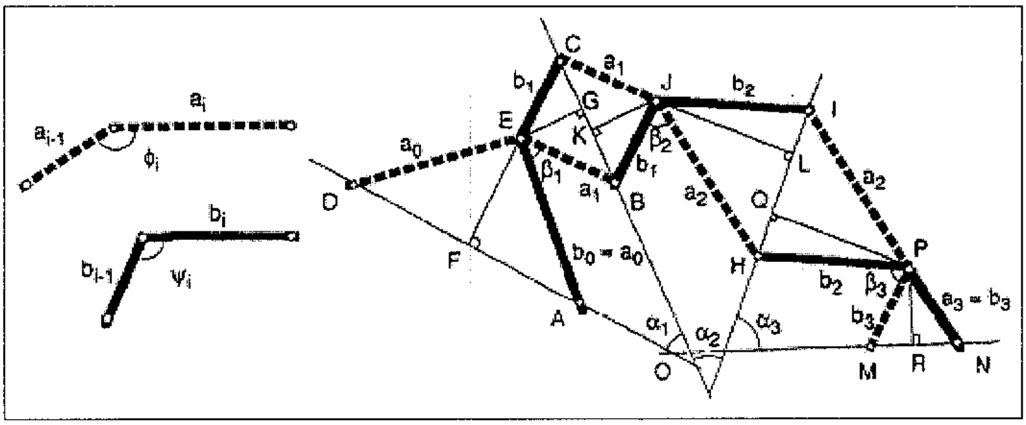


Figure 5 : General Type I GAE

2.2.2.2 *Type II GAE*

Figure 6 below will show the simplest Type II GAE, which is formed by angulated rods with proportional semi-lengths and equal kink angles. This figure has this following relationship

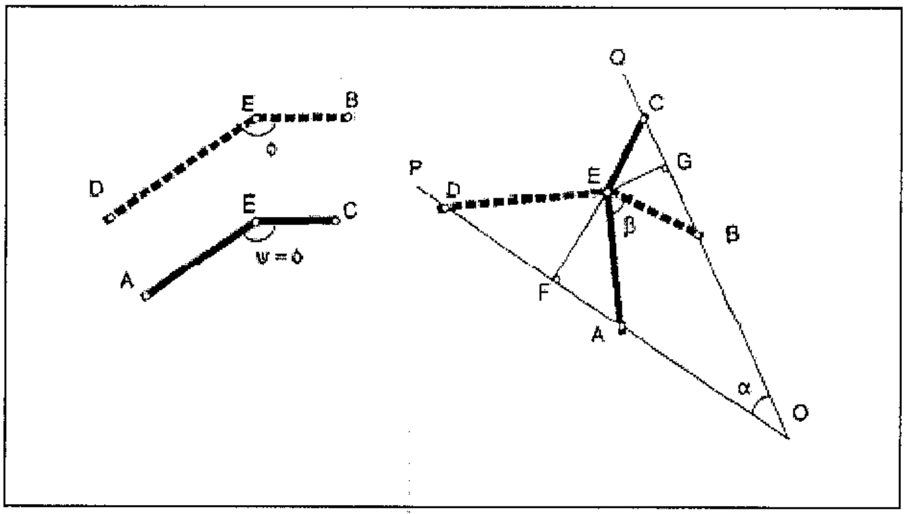


Figure 6 : Simplest Type II GAE

$$\frac{\overline{AE}}{\overline{DE}} = \frac{\overline{CE}}{\overline{BE}}, \quad \text{and} \quad \psi = \phi$$

$$\angle BEG = \angle DEF$$

$$\alpha = 180^{\circ} - \phi$$

Figure 7 is the more general Type II GAE, consisting two similar triangles connected by two parallelograms.

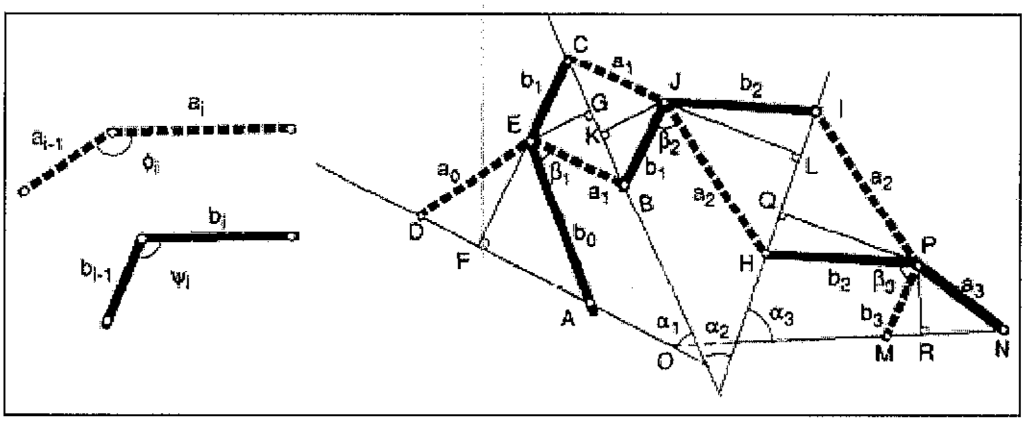


Figure 7 : General Type II GAE

This element satisfies several conditions which indicate that each loop is a parallelogram and the triangles on the sides, triangle AED and triangle NPM are similar. To prove that the angle α is constant, combine some of previous equations and finally would come up of constant equation.

$$x = 3 \times 180^{\circ} - \Sigma\psi = \text{constant}$$

For further explanation, please refer to Appendix A.

2.3 Applications

2.3.1 Architecture

Hoberman Associates have been developing and generated some ideas to use the scissor hinge system onto some structures, in order to deliver its benefits to a structure. In terms of architecture, Hoberman Associates have been developed some structures using this system such as pavilion, plaza, theater and retractable façade. Façade is a decorative band of panels or other materials around the perimeter of the roof, often used to conceal the gable. What Hoberman Associates has created is a wall of a building that can effectively disappear when arch-shaped façade retracts. [9]

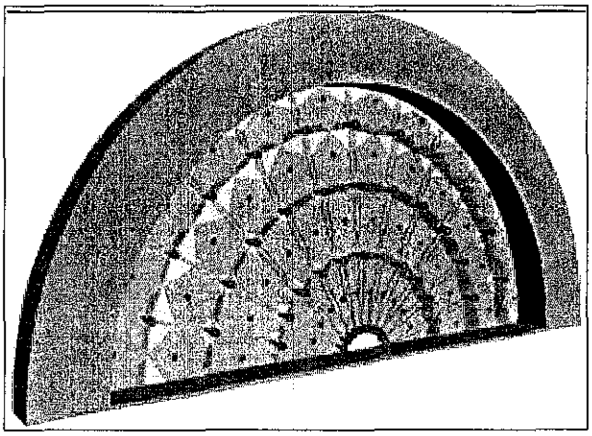


Figure 8 : Retracting façade.

This also applies to a pavilion which is created by connecting two wedge-shaped roof elements together.

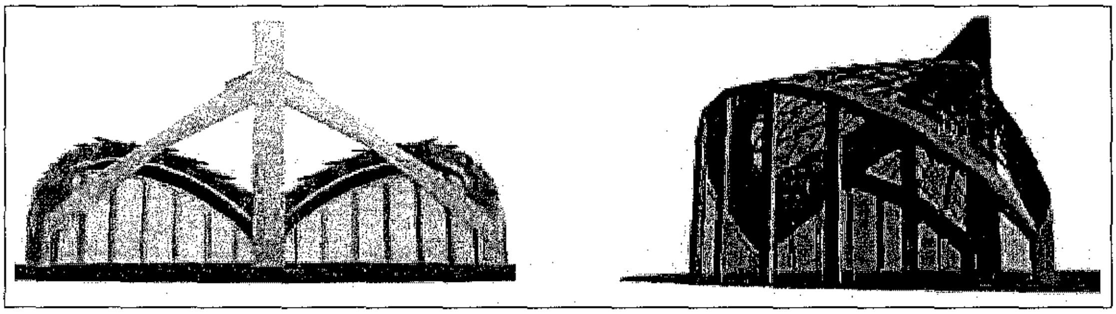


Figure 9 : Retracting pavilion.

Hoberman Associates also redesigned a plaza with a retractable roof element, joined to a high-rise creating a multi-functional space. When the space is covered, the space functions as part of the building lobby. When the roof retracts, the space is opened becoming an extension of the plaza in front of the building.

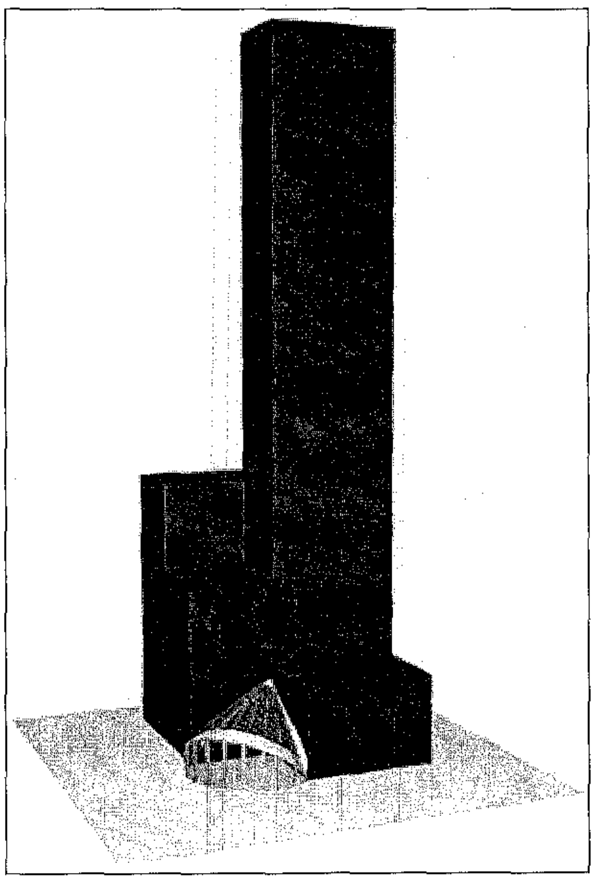


Figure 10 : Retracting roof to create multi-functional space

The retracting element of a roof also extends to provide open-air space for a theater, depending on the weather. The roof widens to cover the theater whenever needed.

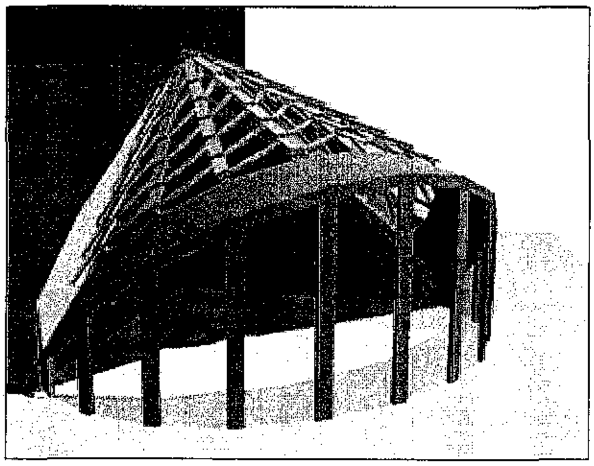


Figure 11 : Theater with retractable roof

2.3.2 Shelter

It has been found that the scissor hinge system could be applied on types shelters, such as tents, or temporary structures like cover for a certain function during the hot weather. More importantly, Hoberman Associates have been developed these types of shelters with mobility, which means, when it is folded neatly, it could be brought anywhere we like, and retract it whenever needed.[9]

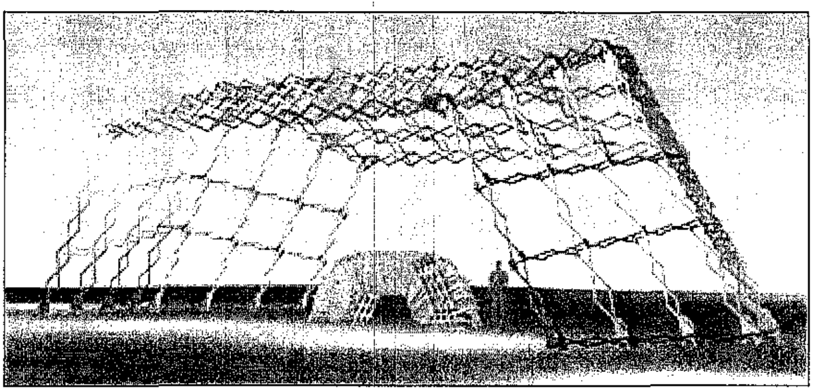


Figure 12 : Retractable shelter

2.3.3 Furniture

In terms of furniture, it has been found that it is possible to make it foldable and retractable. Thus, will save space for storage and easy for mobilization. Furniture can be carried like a briefcase when folded. Pieces can be easily stored on shelves.[9]

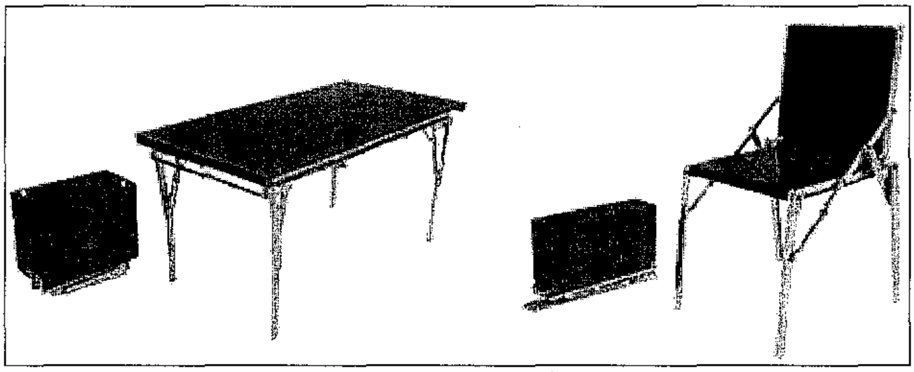


Figure 13 : Portable furniture

2.3.4 Medical

Hoberman Associates has been collaborated with Johnson & Johnson to create a new generation of surgical devices using the retractable scissor hinge system. Transformation technology has many applications within the field of medicine including: tools for non-invasive surgery, expanding stents, and diagnostic devices.[9]

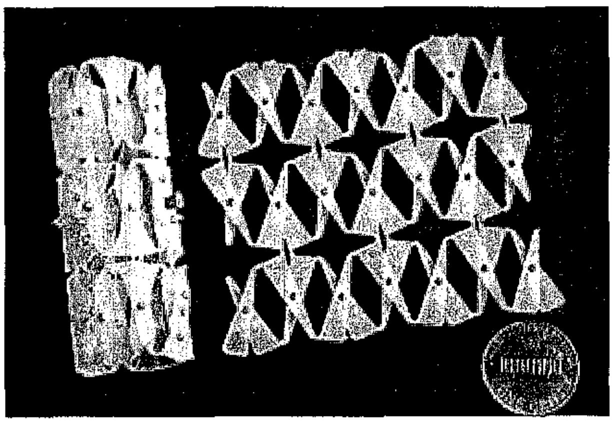


Figure 14 : Retracting surgical devices

2.3.5 Others

Art exhibition has been focused quite a lot by Hoberman Associates. Most of the art sculptures were installed in corporate spaces, science museums, cruise ships and retail environments. They intended to create sculptures that are interactive and exhibits experiential and visually attractive. Hoberman Associates also created toys that have been implying the retracting foldable bar and it has been flourishing to the market.[9]

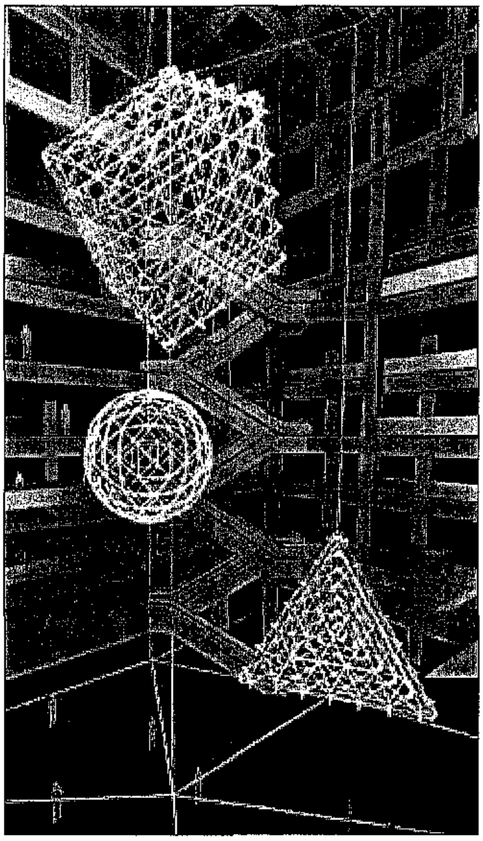


Figure 15 : Retracting Art Sculpture

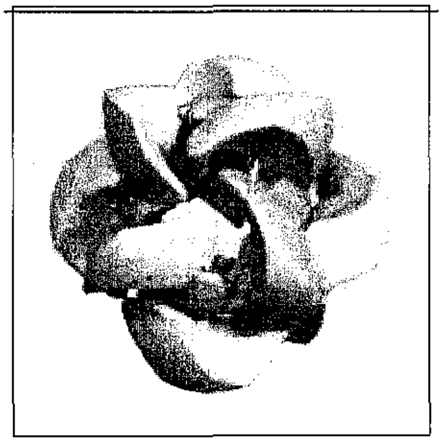


Figure 16 : Retracting toy

CHAPTER 3
METHODOLOGY

This study is divided into three main categories of tasks. First is to understand the concept of foldable bar structure, which has been done during the first month after started this project. Journals and papers written and published by famous civil engineers and Universities Professors are the key of understanding the main concept in this project. The resources that could be found are through the internet and in the library, but most books have only vague explanation on these structures as this concept is still fresh and new. Further study is needed in order to find other alternatives of concept to build such structures. This also includes finding solutions and methods of building the structure, which has been started and applied to mock materials. Hard cardboards have been chosen to be the mock material, replacing the steel bars. In addition to that, thumb tacks are being used as the main of hinge connecting the bars.

The second category of study is to understand and study the engineering behaviour of foldable bar system. This is done to focus on the movement behaviour as these two properties are very critical in designing a structure. This step can be proceeded by using suitable structural analysis software that the University is provided. The software provided for this is the STAADpro software, which is used worldwide. But in order to use this software, one must learn and take some classes to learn how to add data and analyse structure using this structure. Furthermore, internet has been source number one in finding other alternative software to learn how the structure would behave. Auto spreadsheets that have been fully programme for this has been created by experienced engineers. Thus other alternative of software has been downloaded for future reference and usage in case if it is found to be much simpler to understand. Beforehand, preliminary design must be done in order to calculate the angles and dimensions for every type of elements and this is done by creating a table of dimensions and applying it onto AutoCAD. AutoCAD can be used to draw with precise measurement and thus by using this, one might know whether the element designed is stable or not. Also, the movement of the structure can be predicted by looking at the design drawn. This category also covers evaluation of this concept, whether it is applicable to structures and also to find out the integrity of a structure if it was constructed using the scissor hinge system.

Subsequently, the design of one proposed structure, to show the application, will be done by using SolidWorks software, whereby the structure would be illustrated by the animation of the retracting mechanism. The software would also generate the properties of the design, and also other design analysis. For this particular project, retracting wall has been chosen to be design using the mechanism that has been developed.

Lastly, the next step is to explore few other options of structures that can be used to apply the scissor hinge theory. This is done by using the internet as the main source to find the possible structure that can be retractable or deployable, regardless the use of the structure being retracting. Here, the author has chosen retracting wall as to illustrate the application of theories found.

CHAPTER 4
RESULTS AND DISCUSSION

4.1 Part I

In the third step of methodology, it has been stated that the next step was to try to build the structure using several methods and observe how it is going to behave and what the results are. This was also done by using AutoCAD 2004 software to draw the similar structure. By using AutoCAD software, drawing the structure would be 100% accurate. Thus from the drawing obtained from it, it was clear to see that the structure designed is retractable and foldable, just by looking at it. The first method was to use the famous Hoberman's Angulated Element. Stated in Z.You's and S. Pellegrino's journal, this method has come with clear figure for better understanding and also some equations to be calculated in order to design structure using this method. Below is a figure showing the result of the design.

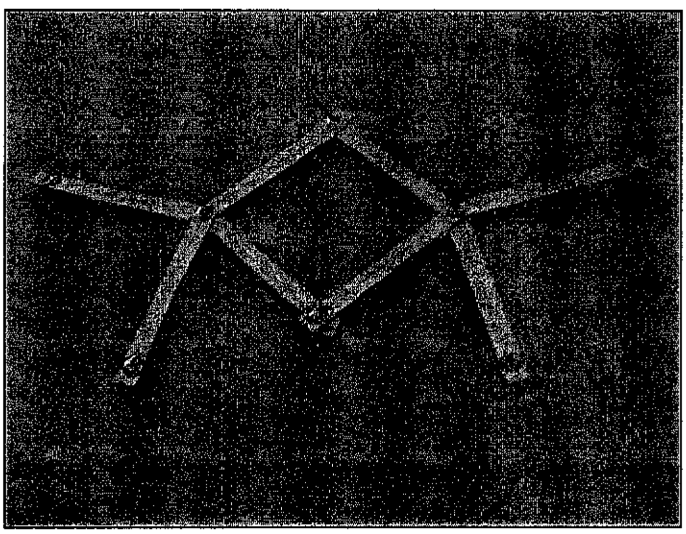


Figure 17 : 1/6 of the whole basic structure using Hoberman's Angulated Element method

Full 360° of structure has been built based on this concept. The concept was proven to precise and accurate as the structure can be fully moved smoothly without fail. However, building this structure only result in basic foldable bar structure. It has yet to visualize how the foldable roof really behaves. The thumb tacks seem to be working well as being the hinges for the structure. But since thumb tacks are usually short, it cannot retain the connection between bars long enough. Other alternative should be explored in order to solve this problem.

The next step that has been taken was unassembled above structure and rearrange it so that the bar is longer, which is, instead of two element in one bar, rearranged it to consist of four (4) elements each foldable bar. Below is the figure of the description.

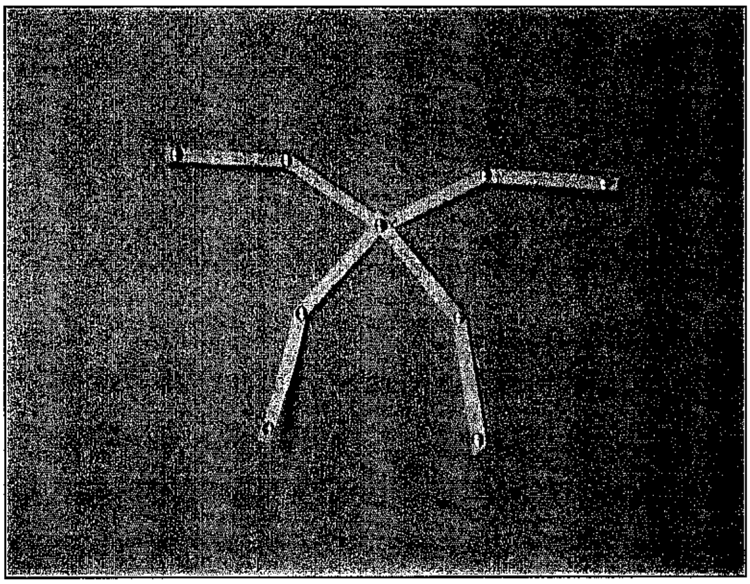


Figure 18 : modifying and experimenting the concept.

The above figure also shows a small part of the whole structure. Unfortunately the picture of the completed structure using this modified concept is yet to be available for the time being. The movement of this structure was running smoothly while being tested. However, the thumb tacks cannot hold the middle connection that long because of its short nail. Even so, it was obvious that the structure was in good condition and proven that the foldable bar can be extended to whatever number of connections depending on one's desire of design.

The last step that has been taken for now was changing the design into longer foldable bars (consist of four elements each bar). Each bar still looks the same, except that instead of having two identical bars and connected to be one, this was fabricated to be only one bar. Figure below was the guideline of all calculations and design made.

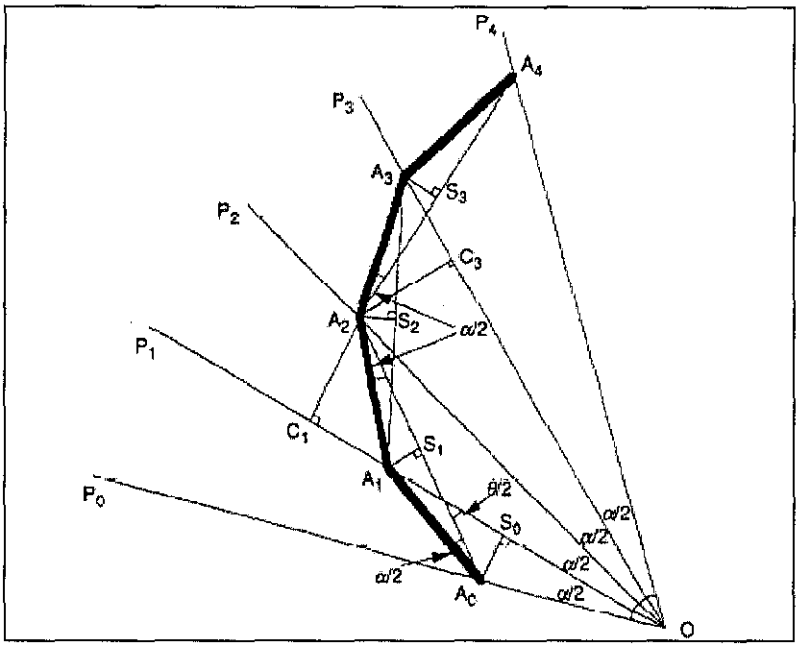


Figure 19 : Multi-angulated Rod

Full 360° of structure based on this bar element has been built. As force is being put to expand and retract, the movement was running unsmooth. The thumb tacks keep falling out and had to reassemble the structure each time. This has been a very tedious experiment thus material improvement is very much in need to replace the hinges. Perhaps an alternative for this is to use small screws but this is yet to be proven effective.

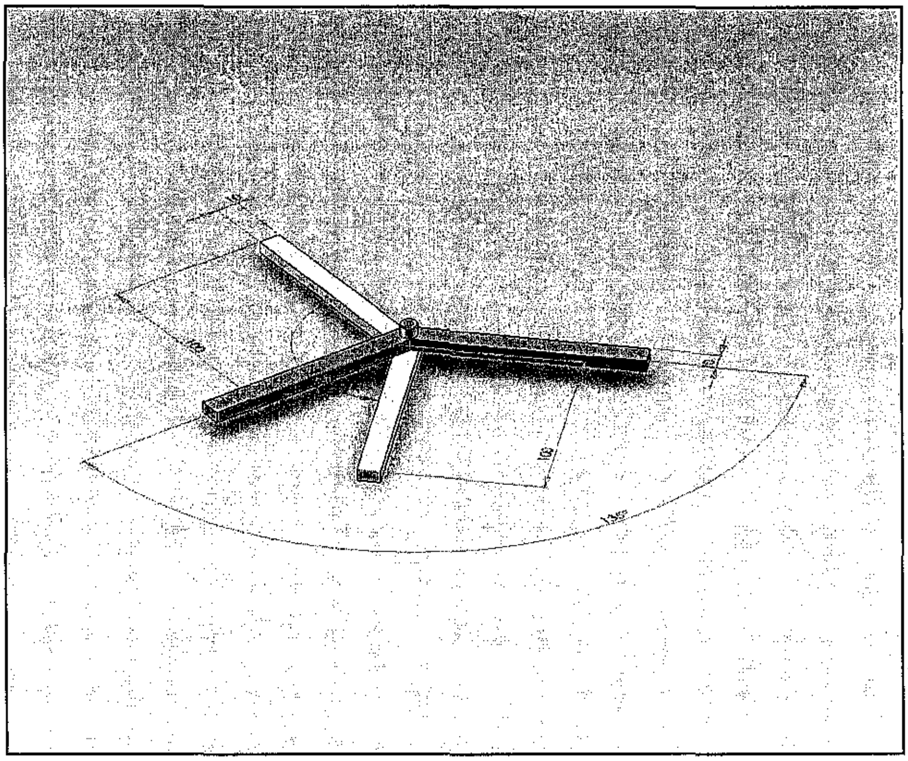
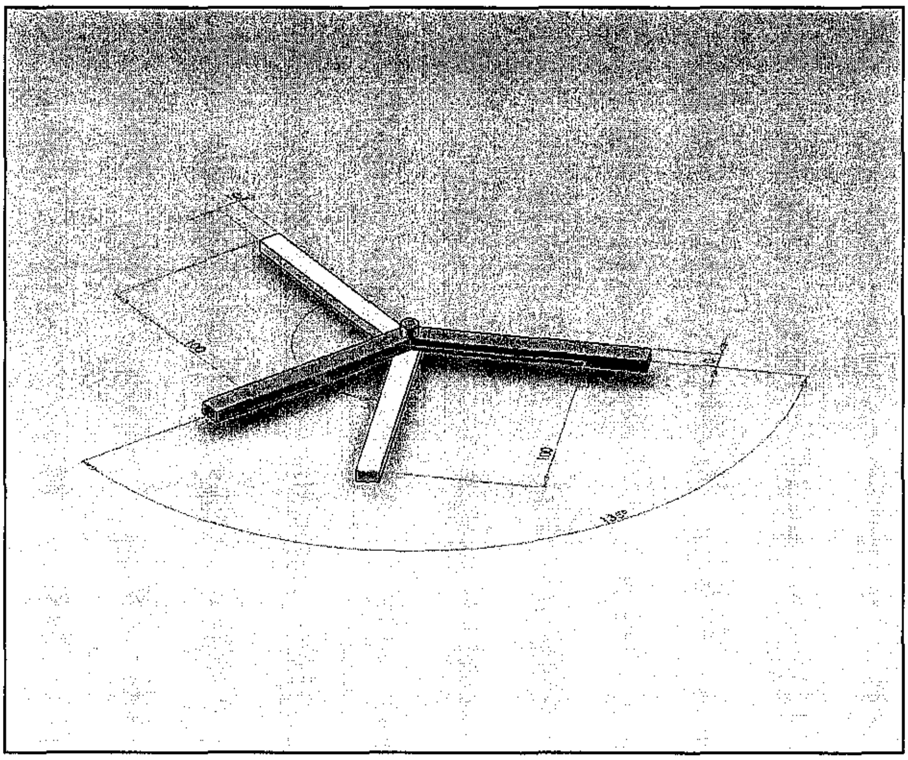


Figure 21 : Isometric view of designed Hoberman's Angulated element

Trial and error method has been the only way to figure out how to design the retracting wall. Below would be the illustration of using Hoberman's Angulated element as the basis of the design and one fixed point has been determined. However, the design had some faulty; the retracting mode cannot be progressed as the fixed point has been chosen incorrectly. The properties for the design has been attached to the Appendix.



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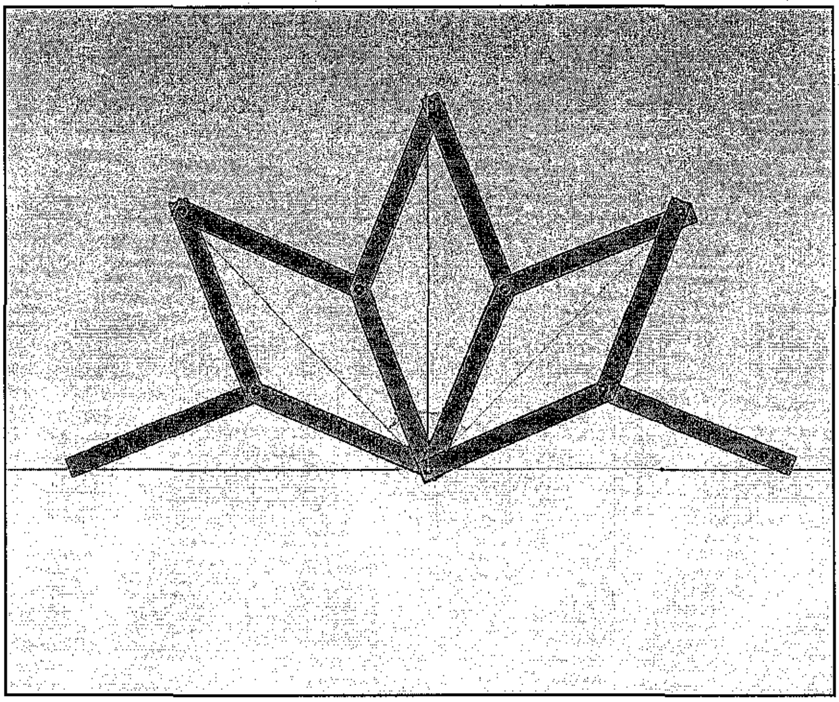

Figure 22 : Faulty design using Hoberman's Angulated Element

4.2.2 Type I GAE

After designing the wall structure using Hoberman's Angulated element, type I GAE element has been used to be the basis for the next design. Below is the designed element that has been employed.

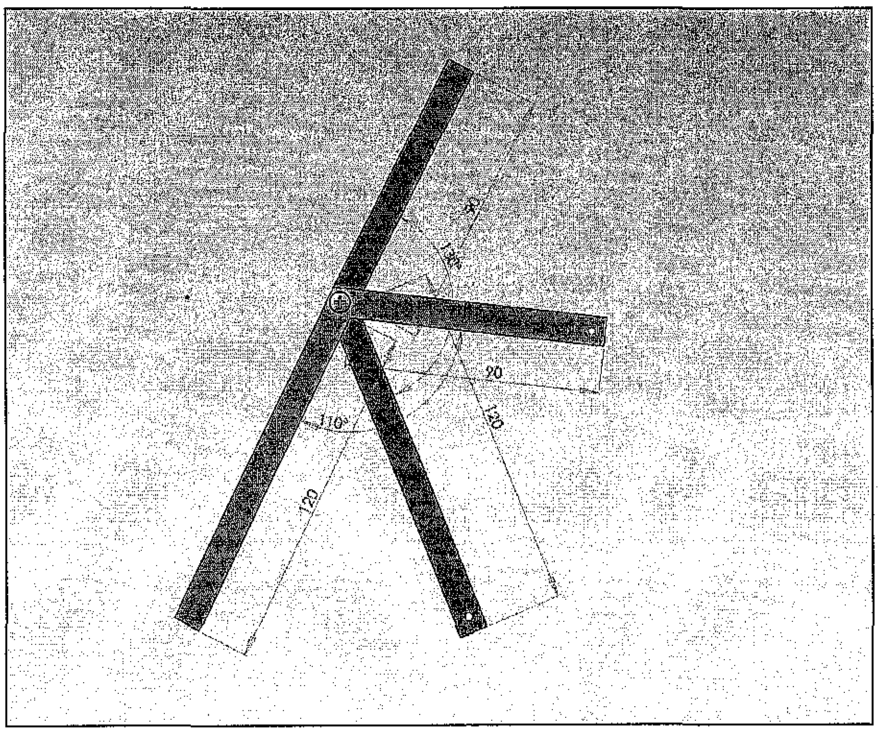

Figure 23 : Front view of designed Type I GAE component

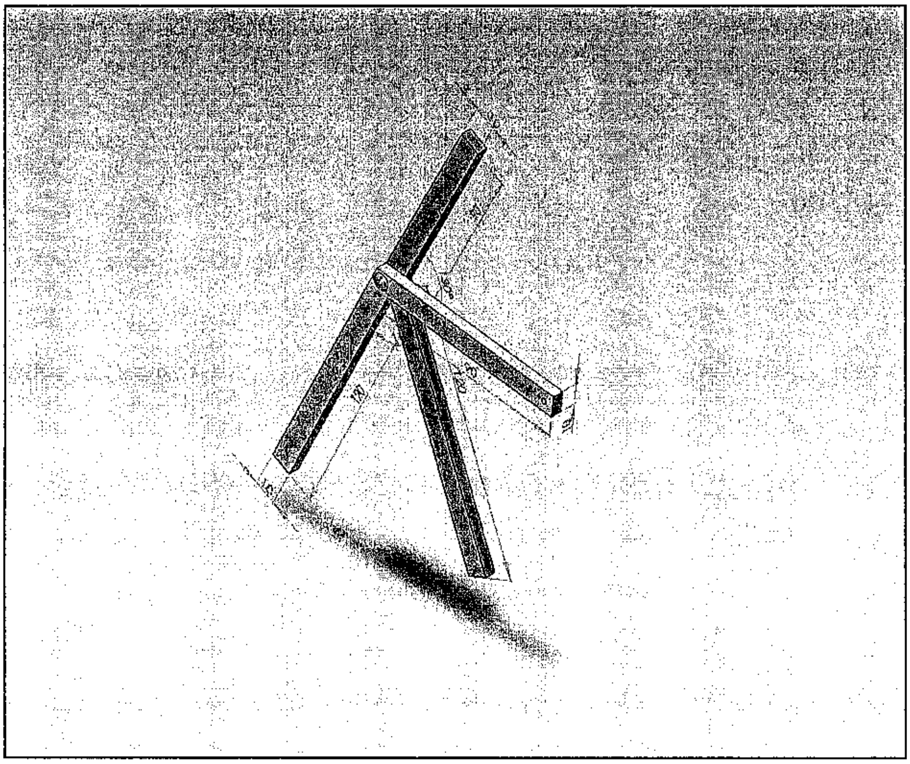

Figure 24 : Isometric view of Type I GAE component

Using trial and error method, the wall structure was designed through several assumptions of dimensions and angles. Below is the figure that will illustrate the design.

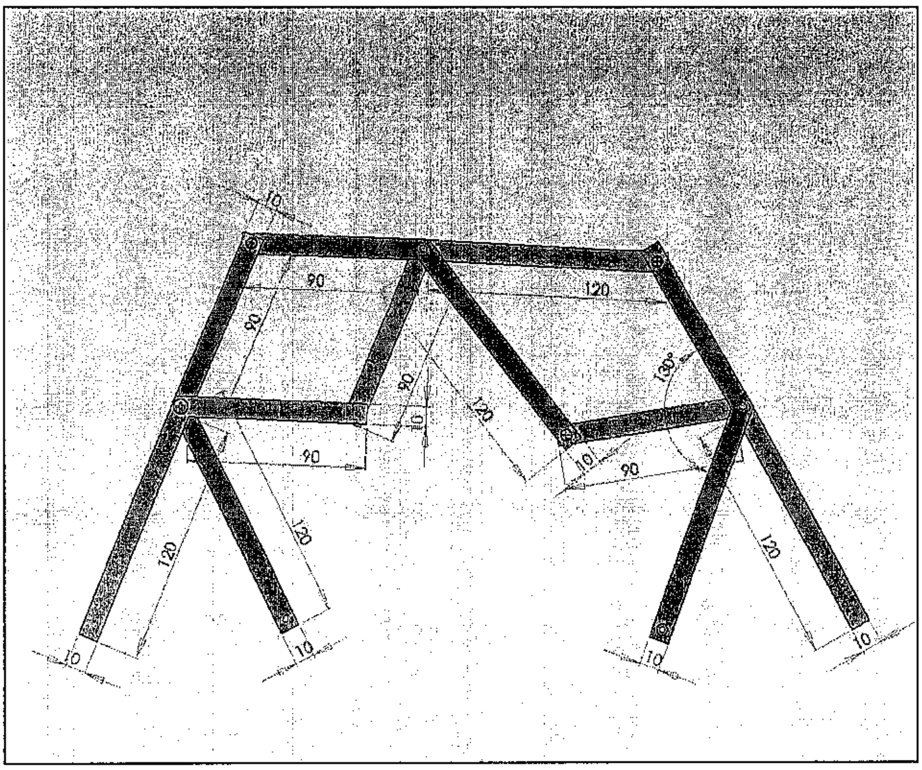

Figure 25 : Retractable wall structure using Type I GAE

Referring to the previous figure, the outcomes should have consists of two (2) parallelograms before and after it has been retracted. However due to some unsolved problems with SolidWorks, the structure has not been adjusted. Thus troubleshooting is needed in order for it to fulfill every criteria of the design; two (2) parallelograms with two (2) isosceles triangles.

4.2.3 Type II GAE

For further study, type II GAE has also been the basis of the proposed retractable wall structure.

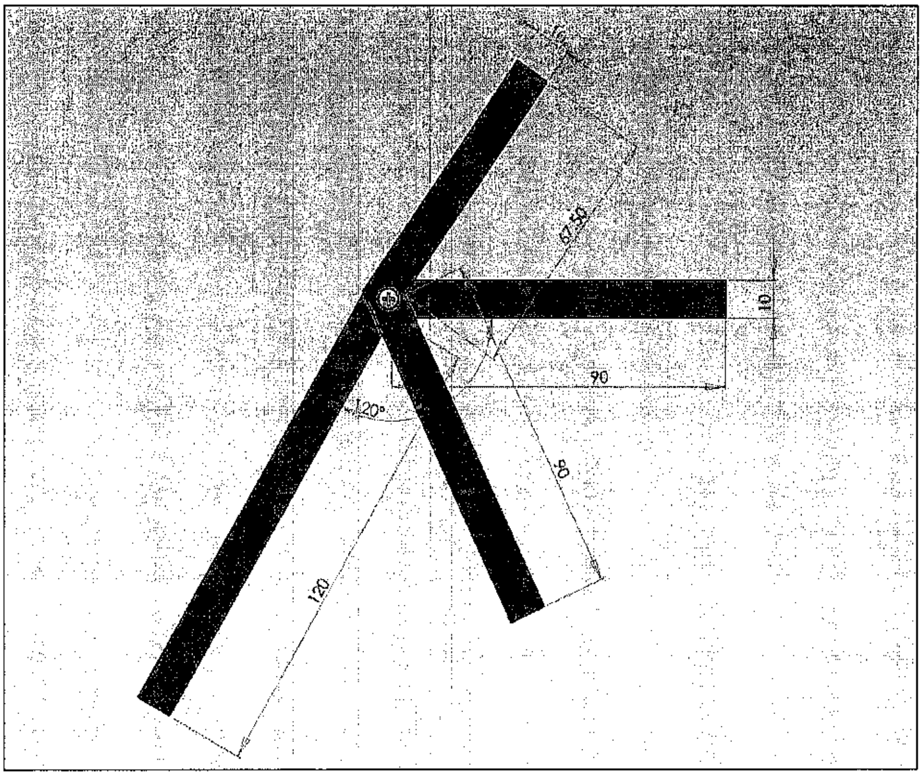

Figure 26 : Front view of Type II GAE component

Figure below is isometric view of the component where you can see the assumed thickness of element.

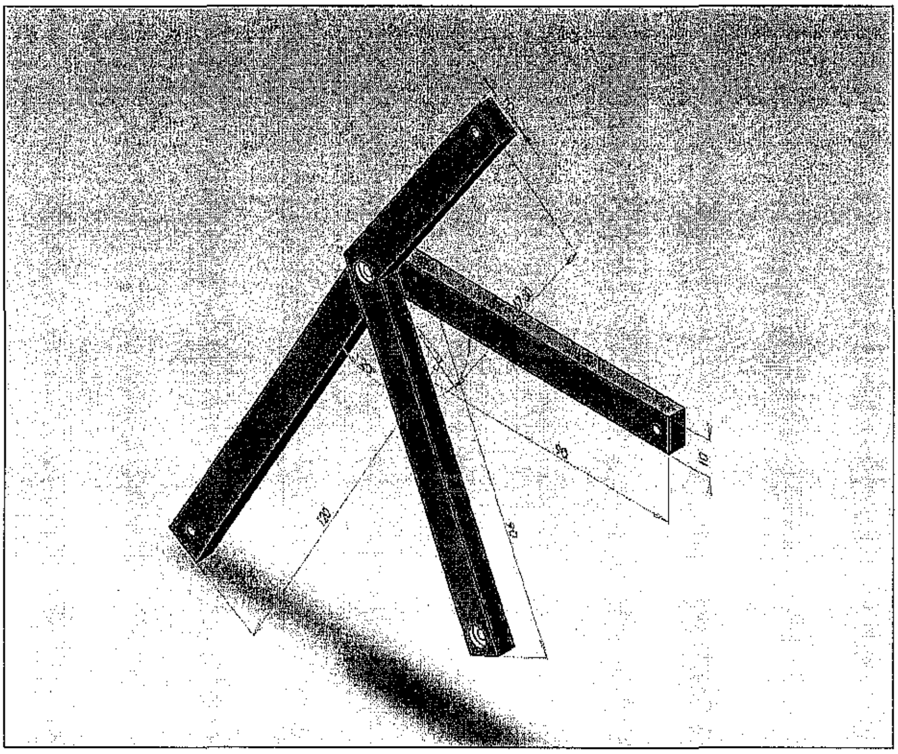

Figure 27 : Isometric view of Type II GAE component

Again, by using trial and error method with some assumptions of dimensions and angles, the outcome of wall structure using type II GAE is illustrated as below.

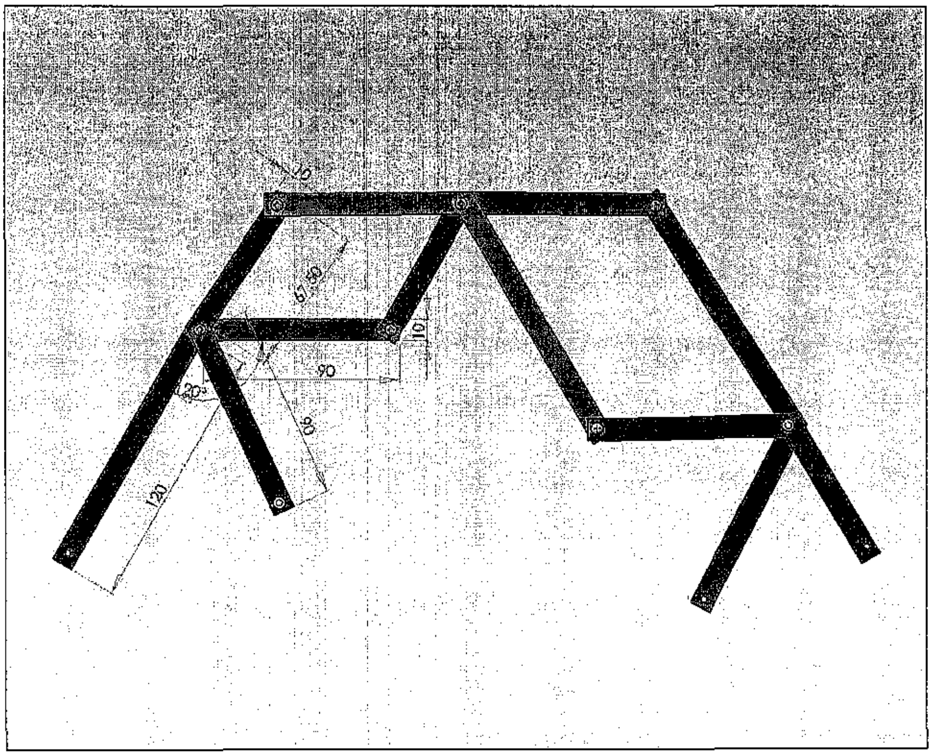

Figure 28 : Retractable wall structure using Type II GAE

Note that for all three basis of design is based on own assumptions on dimensions and angles of retracting. This can be referred to Appendix B.

CHAPTER 5
CONCLUSION AND RECOMMENDATIONS

5.1 Conclusion

Since the first day that this task is being given, many basic retractable structures have been built based on several concepts stated in journals found through the internet. For final year project part I, the author was successfully understand every theory behind each different element that has been simplified in selected engineers' journals. The author had also done some experimenting with the elements by using certain materials in order to justify the correctness of the theories. Different dimensions of elements have also been implied to the element to see if there will be any effects taken upon. Thus the main objectives which are to study and evaluate the mechanisms have been achieved. Although, the analysis of the structure in terms of stress and strain is yet to be done but have been experimenting the structure by modifying the concepts and theories. The structures successfully retracted and closed, even though there are some complications on running smoothly. For part two, a new scope of work has been expanded; pre-design a wall structure using retractable mechanism of foldable bar structure. SolidWorks has been a good assistance for the design and thus a few detail drawings have been created. Some of the design are faulty, due to the only method that is being used; trial and error method. The simulation of the design has also been created but unfortunately cannot be shown here. From the simulation, it has been obvious that the structure can be retracted smoothly despite the fact that the design created has been only pre-designed with inaccurate measurement. Further expansion of this study is needed by other individual.

5.2 Future Work for Expansion and Continuation

It would be interesting if any individual would continue this work with further study on how the thickness of previous element would affect the retracting mechanism. Also, the detail study of 3D structure with this 2D mechanism relationship; how the joints are made and the criteria. Due to unforeseen time constraint, the author has been unable to come up with design limitations for the retractable element. Thus it would be appealing if any individual could continue the author's work with experiments to find out the design limit and further design guideline as the references on this study was adequate.

Foldable bar is not the only option for retractable mechanism. The author believes that there are many other alternatives to the retracting mechanism. The author suggested to continue study on other retracting mechanism.

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APPENDIX A

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FOLDABLE BAR STRUCTURES

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Abstract—A new, general type of two-dimensional foldable structures is presented, which extends and generalises the standard trellis-type foldable structure consisting of two sets of parallel *straight rods* connected by hinges. It is shown that any structure consisting of rigid, *multi-angulated rods*, i.e., straight rods with kinks at the hinge positions, can be folded if the rods form a tessellation of parallelograms. This discovery is exploited to investigate the structural layouts of flat and curved structures which can be folded along their perimeter. © 1997 Elsevier Science Ltd.

1. INTRODUCTION

A simple, two-dimensional foldable structure can be made from two sets of parallel, straight rods connected by pivots, or *scissor hinges*, at all intersection points. A scissor hinge is a revolute joint whose axis is perpendicular to the plane of the structure. Both structures shown in Fig. 1 are of this type, and both can be folded by freely deforming their cells in shear until the two sets of rods become approximately parallel, for $\theta \cong 0^\circ$ or $\theta \cong 180^\circ$. During folding, each set of collinear pivots remains collinear, and all pivots become collinear—in theory, at least—in the fully-folded configuration. This is the principle behind many commonly used foldable structures, e.g. garden trellises, wine racks, awnings, etc. The same concept has already been used for more exotic applications, such as movable theatre structures (Pinero, 1961, Escrig, 1993), but it affords only limited freedom to engineers designing structures whose shape, boundaries, etc. are already specified. Hence, numerous efforts have been made to find more general solutions.

Broadly speaking, there are two different ways of extending this simple, intuitive concept to more general shapes. One option is to look for a repeating building block with an internal degree of freedom which allows folding; another option is to design a complete structure, whose shape is determined mainly by its particular application, and then to modify its geometry, member properties and layout, connections, etc. until the structure can be folded and deployed without damage, albeit with some elastic deformation of its members.

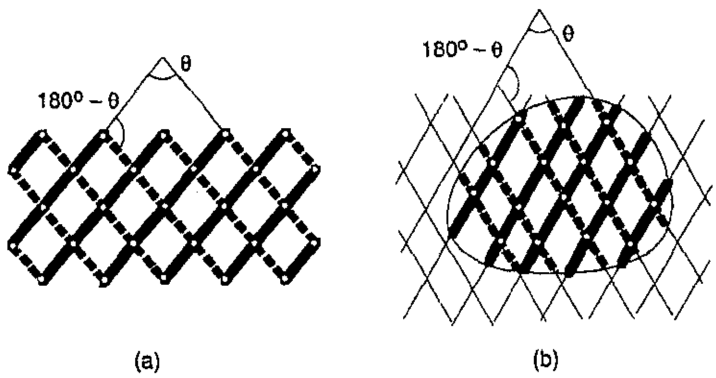


Fig. 1. Trellis-type foldable structures formed by two sets of straight, parallel rods.

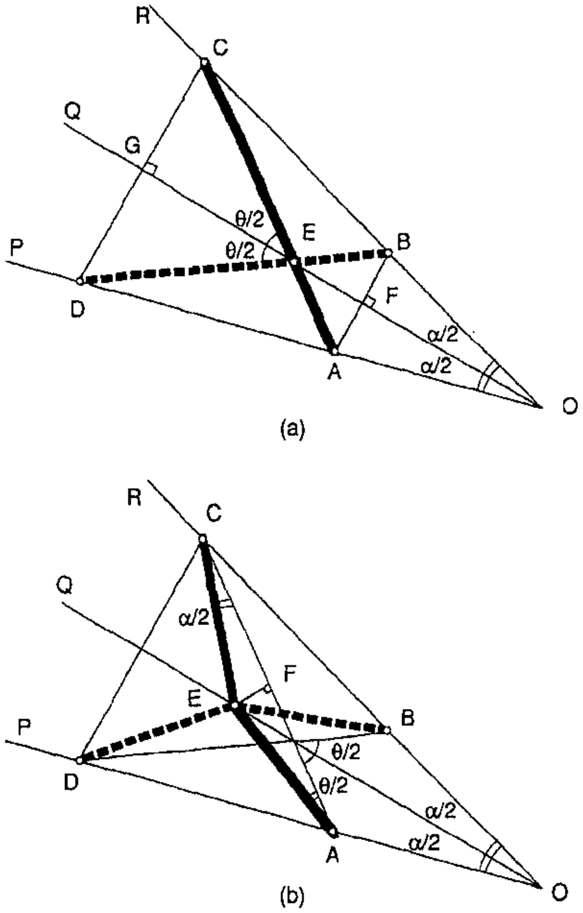


Fig. 2. (a) Ordinary pantographic element consisting of straight rods. (b) Symmetric angulated element with a constant angle of embrace and $\overline{AE} = \overline{BE} = \overline{CE} = \overline{DE}$.

The advantage of the first, modular approach is that, once a suitable building block has been found, then a whole class of foldable structures may become available simply by changing the number and size of the blocks. Two- and three-dimensional assemblies of pairs of straight bars connected by scissor hinges, which form single-degree-of-freedom mechanisms (Clarke, 1984, Escrig, 1985), have been used as building blocks for many complex structural mechanisms (Zanardo, 1986, Kwan and Pellegrino, 1991). However, it is not generally true that a structure consisting of foldable modules is always foldable: it is also required that the interfaces between modules deform in a compatible fashion, and in some cases there may also be one or more global conditions that need to be satisfied (Pellegrino and You, 1993, You and Pellegrino, 1994).

A recent development, of crucial importance to this paper, was the invention of the simple *angulated element* (Hoberman, 1990), consisting of a pair of identical angulated rods connected together by a scissor hinge, see Figs 2 and 3. In analogy with elements made from straight rods, which—under certain conditions—fold while maintaining the end pivots on parallel lines, angulated elements subtend a constant angle as their rods rotate. This property is exploited in Hoberman's foldable sculptures (Waters, 1992) and in Servadio's foldable polyhedra (1994). Among the more practical applications of simple angulated elements is the Iris Retractable Roof (Hoberman, 1991), a foldable dome with circular plan consisting of concentric rings connected by scissor hinges.

This approach to foldable bar structures has already produced many ingenious solutions, but real, large scale applications have yet to follow. A key disadvantage of this approach is that any solution is, in a sense, unique, i.e., valid only for a particular shape of structure and for a specific set of boundary conditions. Furthermore, the number of solutions currently known is quite small. For example, when designing the plan shape of

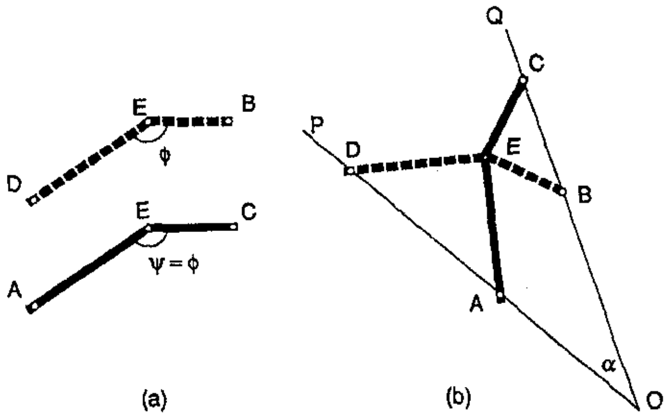


Fig. 3. General type of Hoberman's element, formed by *identical* angulated rods: $\overline{AE} = \overline{DE}$, $\overline{BE} = \overline{CE}$, $\psi = \varphi$. ADE and BCE are similar isosceles triangles.

the Iris Retractable Roof there are only three free parameters, which are the total number of rings, the number of angulated elements in each ring, and the length of the angulated rods. Everything else follows from geometric rules.

The second, global approach to the design of foldable bar structures has been pioneered by Pinero (1961) and Zeigler (1981, 1984, 1987, 1993) and has been further developed by Escrig *et al.* (1989) and Escrig and Valcarcel (1993). The idea is to design foldable structures by a two-stage process. First, the overall geometry of the structure is decided, such that all members are unstrained both in the required, fully deployed configuration and also in the folded configuration, usually a compact bundle where all bars are theoretically parallel. Second, a detailed structural analysis of the folding process is carried out, to check that any strains induced by the folding process are sufficiently small (Gantes *et al.*, 1991). Most foldable structures based on this approach consist of pairs of straight rods connected by off-centre scissor hinges. In general, the achievement of satisfactory behaviour during folding is at the expense of low deployed stiffness, and hence locking elements are usually incorporated in structures of this type. Although, in principle, any solution can be modified to suit the requirements of a particular application, even small changes will require some re-analysis.

We have recently discovered (You and Pellegrino, 1996) a new approach to foldable bar structures which combines the key advantages of the two approaches described above. This new approach makes use of a new, large family of foldable building blocks, which we call *generalised angulated elements*. These elements subtend a constant angle during folding, as Hoberman's simple angulated element, but afford much greater freedom than all other elements that have been used previously. We have also discovered that a series of contiguous angulated rods can be replaced with a single, *multi-angulated rod*, which is an extension of the straight rod with collinear pivots used in the simple foldable structures shown in Fig. 1, thus reducing significantly the number of component parts of a structure and the complexity of its joints. These two discoveries open up a range of new applications for large scale, foldable bar structures.

Although our discoveries are not restricted to a particular type of application, the examples that are presented in the paper are aimed towards foldable roof structures for stadia, swimming pools, etc. (Levy, 1995).

The layout of the paper is as follows. Section 2 briefly reviews the derivation of the simple angulated element. Section 3 introduces the new generalised angulated elements (GAE's), which consist of chains of any number of parallelograms, connected to adjacent elements either by two isosceles triangles, or by two similar triangles. The special properties of symmetric GAEs are discussed. Section 4 deals with assemblies of simple angulated elements that form circular, rotationally symmetric foldable structures. Starting from the solution originally proposed by Hoberman, consisting of separate angulated elements connected by scissor hinges, it is shown that a geometrically identical, but more efficient foldable structure can be made from multi-angulated elements. Section 5 deals with foldable structures of general shape, consisting of generalised angulated elements. Several configurations for foldable roof structures which can be folded along their perimeter, and have different shape, e.g. rectangular, elliptical, etc., are presented. Section 6 shows that three-dimensional variants of the solutions obtained in the paper can be obtained by projecting any of these layouts onto a three-dimensional surface. Double-layer foldable structures are also obtained in a similar way. Section 7 concludes the paper.

2. HOBERMAN'S ANGULATED ELEMENT

Figure 2(a) shows an ordinary pantographic element, made of a pair of *identical straight rods*, hinged together by a scissor hinge at E. The end connectors, A, B, C, and D define two straight lines OP and OR. The element is symmetric with respect to the line OQ. A relationship between α, the angle subtended by this element, and θ, the *deployment angle*, can be obtained by noting that

$$\overline{CG}-\overline{BF} = \overline{FG}\tan\alpha/2 \tag{1}$$

where

$$\overline{CG} = \overline{CE}\sin\theta/2 \tag{2}$$

$$\overline{BF} = \overline{BE}\sin\theta/2 = \overline{AE}\sin\theta/2 \tag{3}$$

and

$$\overline{FG} = \overline{AC}\cos\theta/2. \tag{4}$$

Substituting eqns (2)–(4) into eqn 1

$$(\overline{CE}-\overline{AE})\sin\theta/2 = \overline{AC}\cos\theta/2\tan\alpha/2 \tag{5}$$

or

$$\tan\alpha/2 = \frac{\overline{CE}-\overline{AE}}{\overline{AC}}\tan\theta/2. \tag{6}$$

It is obvious from eqn (6) that α varies with θ. Supposing that the positions of OP and OR are fixed, it can be concluded that it is impossible to mobilise the pantographic element ABCD, i.e., to vary the angle θ, if A, D and B, C are allowed to move only along the lines OP and OR (Zanardo, 1986).

This difficulty can be resolved (Hoberman, 1990, 1991) by using non-straight, "angulated" rods, and hence by moving the pivot E to the new, special position shown in Fig. 2(b). It can be readily shown that

$$\tan\alpha/2 = \frac{\overline{CF}-\overline{AF}}{\overline{AC}}\tan\theta/2+2\frac{\overline{EF}}{\overline{AC}}. \tag{7}$$

For $\overline{AF} = \overline{CF}$, i.e., F half way between A and C, the first term on the right-hand-side vanishes. Hence, α becomes a constant for all θs, and it is now possible to mobilise the pantographic element with A, D and B, C lying, respectively, on the lines OP and OR.

Therefore, deployment requires that the following two conditions be satisfied

$$\overline{AF} = \overline{CF} \quad \text{and} \quad \alpha = 2\arctan\frac{\overline{EF}}{\overline{AF}}. \tag{8}$$

From eqn (8), it can be shown that

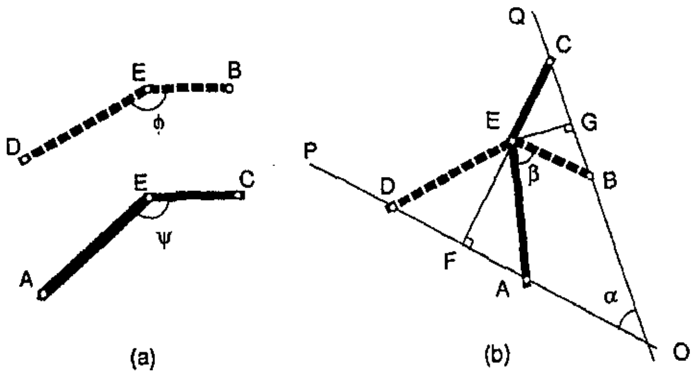


Fig. 4. Simplest Type I GAE, formed by angulated rods with equal semi-length but different kink angles $\overline{AE} = \overline{DE}$, $\overline{BE} = \overline{CE}$, $\psi \neq \varphi$. ADE and BCE are isosceles triangles.

$$\angle CAE = \angle ACE = \alpha/2 \tag{9}$$

$$\angle AEC = 180° - \alpha \tag{10}$$

and hence the distance between an end hinge and the internal hinge is a constant

$$l = \sqrt{\overline{AF}^2 + \overline{EF}^2}. \tag{11}$$

Hoberman (1990, 1991) has shown that the above derivation can be extended to non-symmetric angulated elements, which are still made of *identical angulated rods*. Figure 3 shows the most general element considered by Hoberman. It satisfies the following conditions

$$\overline{AE} = \overline{DE}, \quad \overline{BE} = \overline{CE}, \quad \text{and} \quad \psi = \phi = 180° - \alpha \tag{12}$$

and hence the angulated rods form two *isosceles triangles*. Note that $\overline{AE}$ is not necessarily equal to $\overline{BE}$.

3. GENERALISED ANGULATED ELEMENTS

A generalised angulated element (GAE) is a set of interconnected angulated rods that form *a chain of any number of parallelograms with either isosceles triangles* (Type I GAE) or *similar triangles* (Type II GAE) *at either end*. A generalised angulated element embraces a constant angle as the element is folded or expanded. Separate proofs of the angles of embrace of Type I and Type II GAEs are given next.

GAEs without any parallelograms are considered first, for simplicity, and it is shown that Hoberman's simple angulated element is a special case of both Type I and Type II GAEs.

3.1. *Type I GAE*

Before discussing the general Type I GAE, consider first the angulated element shown in Fig. 4, which has

$$\overline{AE} = \overline{DE}, \quad \overline{BE} = \overline{CE} \quad \text{and, in general,} \ \psi \neq \phi. \tag{13}$$

From Fig. 4, the sum of the internal angles in the quadrangle OGEF is 360° and, since $\angle OFE = \angle OGE = 90°$, the angle α can be expressed as

$$\alpha = 180° - (\angle AEF + \beta + \angle BEG). \tag{14}$$

Because $\triangle ADE$ and $\triangle BCE$ are isosceles triangles,

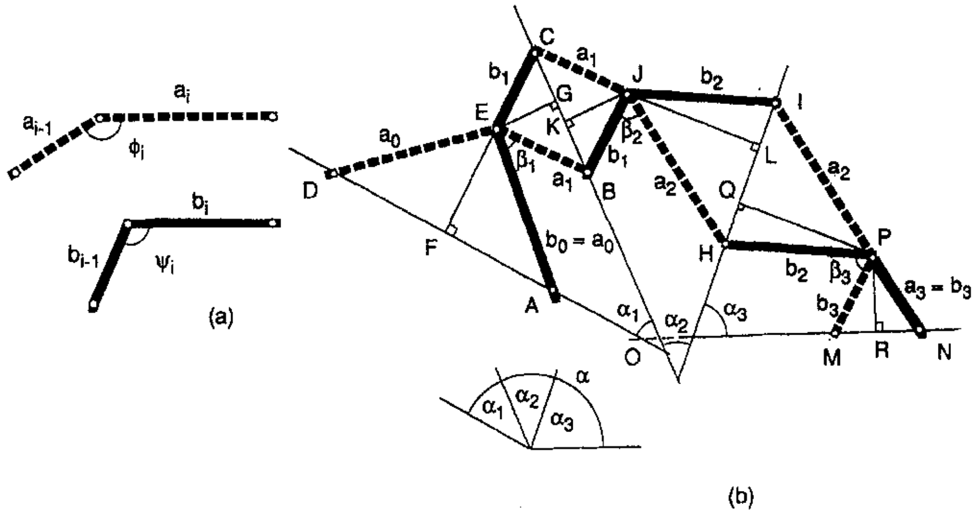


Fig. 5. General Type I GAE consisting of two isosceles triangles connected by two parallelograms.

$$\angle \mathrm{AEF} = \frac{\phi - \beta}{2} \quad \text{and} \quad \angle \mathrm{BEG} = \frac{\psi - \beta}{2}. \tag{15}$$

Hence, substituting eqn (15) into eqn (14)

$$\alpha = 180° - \frac{\phi + \psi}{2} = \text{constant} \tag{16}$$

which shows that this element subtends a constant angle. Note that Hoberman's element is re-obtained, when $\phi = \psi$.

A most interesting special case is obtained when either $\phi = 180°$, or $\psi = 180°$, which implies that one rod is angulated, while the other rod is straight.

More general Type I GAEs are made from two or more angulated elements. Figure 5 shows an example with three elements, which satisfy the following conditions:

(i) each closed loop is a parallelogram, i.e.,

$$\overline{\mathrm{CE}} = \overline{\mathrm{BJ}} \quad \text{and} \quad \overline{\mathrm{EB}} = \overline{\mathrm{CJ}}, \quad \overline{\mathrm{HJ}} = \overline{\mathrm{IP}} \quad \text{and} \quad \overline{\mathrm{IJ}} = \overline{\mathrm{HP}}. \tag{17}$$

(ii) $\triangle$AED and $\triangle$NPM are isosceles triangles, i.e.,

$$\frac{\overline{\mathrm{DE}}}{\overline{\mathrm{AE}}} = \frac{\overline{\mathrm{MP}}}{\overline{\mathrm{NP}}} = 1. \tag{18}$$

Note that the structure shown in Fig. 5 can be regarded as being formed by "cutting" the element shown in Fig. 4 at the scissor hinge E and inserting parallelograms in between the triangles formed thus.

Next, it will be shown that the angle α embraced by this element has constant magnitude. From Fig. 5, it can be obtained that

$$\alpha = \angle \mathrm{DON} = \alpha_1 + \alpha_2 + \alpha_3 \tag{19}$$

where

$$\alpha_1 = 180° - (\angle \mathrm{AEF} + \beta_1 + \angle \mathrm{BEG})$$
$$\alpha_2 = 180° - (\angle \mathrm{BJK} + \beta_2 + \angle \mathrm{HJL})$$

$$\alpha_3 = 180^\circ - (\angle\mathrm{HPQ} + \beta_3 + \angle\mathrm{MPR}). \tag{20}$$

Condition (i) implies

$$\angle\mathrm{BEC} = \angle\mathrm{BJC} \quad \text{and} \quad \angle\mathrm{HJI} = \angle\mathrm{HPI} \tag{21}$$

and also

$$\angle\mathrm{BEG} + \angle\mathrm{BJK} = \angle\mathrm{BEC} \quad \text{and} \quad \angle\mathrm{HJL} + \angle\mathrm{HPQ} = \angle\mathrm{HJI}. \tag{22}$$

Substituting eqns (20), (21) and (22) into eqn (19) gives

$$\alpha = 3 \times 180^\circ - (\angle\mathrm{AEF} + \psi_1 + \psi_2 + \beta_3 + \angle\mathrm{MPR}). \tag{23}$$

From condition (ii), we know that

$$\angle\mathrm{AEF} = \frac{\angle\mathrm{AED}}{2} = \frac{\phi_1 - \beta_1}{2} \quad \text{and} \quad \angle\mathrm{MPR} = \frac{\angle\mathrm{MPN}}{2} = \frac{\psi_3 - \beta_3}{2}. \tag{24}$$

Note that eqn (21) can be rewritten as

$$\psi_1 - \beta_1 = \phi_2 - \beta_2 \quad \text{and} \quad \psi_2 - \beta_2 = \phi_3 - \beta_3 \tag{25}$$

and adding up these two equations gives

$$\psi_1 + \psi_2 - \beta_1 = \phi_2 + \phi_3 - \beta_3. \tag{26}$$

Adding $\phi_1 + \psi_3$ to both sides of eqn (26) and tidying up gives

$$\phi_1 - \beta_1 = \Sigma\phi - \Sigma\psi + \psi_3 - \beta_3 \tag{27}$$

where

$$\Sigma\phi = \phi_1 + \phi_2 + \phi_3 \quad \text{and} \quad \Sigma\psi = \psi_1 + \psi_2 + \psi_3. \tag{28}$$

Substituting eqn (27) into eqn (24), and the result into eqn (23) gives

$$\alpha = 3 \times 180^\circ - \frac{\Sigma\phi + \Sigma\psi}{2} = \text{constant}, \tag{29}$$

i.e., the angle of embrace of a Type I GAE is a constant.

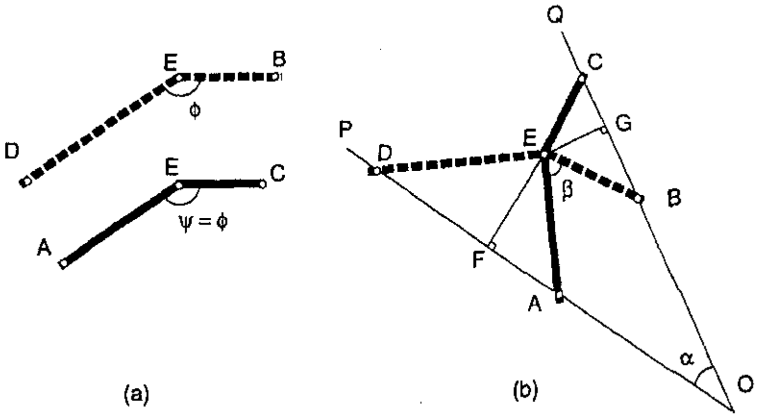


Fig. 6. Simplest Type II GAE, formed by angulated rods with proportional semi-lengths and equal kink angles $\overline{AE}/\overline{DE} = \overline{CE}/\overline{BE}$, $\psi = \varphi$. ADE and BCE are similar triangles.

3.2. *Type II GAE*

Consider first the angulated element shown in Fig. 6, which has

$$\frac{\overline{AE}}{\overline{DE}} = \frac{\overline{CE}}{\overline{BE}}, \quad \text{and} \quad \psi = \phi. \tag{30}$$

To show that the angle α is constant in this case, we note that eqn (14) is still valid. Because $\triangle$AED and $\triangle$BEC are similar,

$$\angle \mathrm{BEG} = \angle \mathrm{DEF} \tag{31}$$

and, substituting eqn (31) into eqn (14)

$$\alpha = 180^\circ - \phi. \tag{32}$$

Note that Hoberman's element is re-obtained when

$$\frac{\overline{AE}}{\overline{DE}} = \frac{\overline{CE}}{\overline{BE}} = 1. \tag{33}$$

A more general Type II GAE is shown in Fig. 7. This element satisfies the following conditions

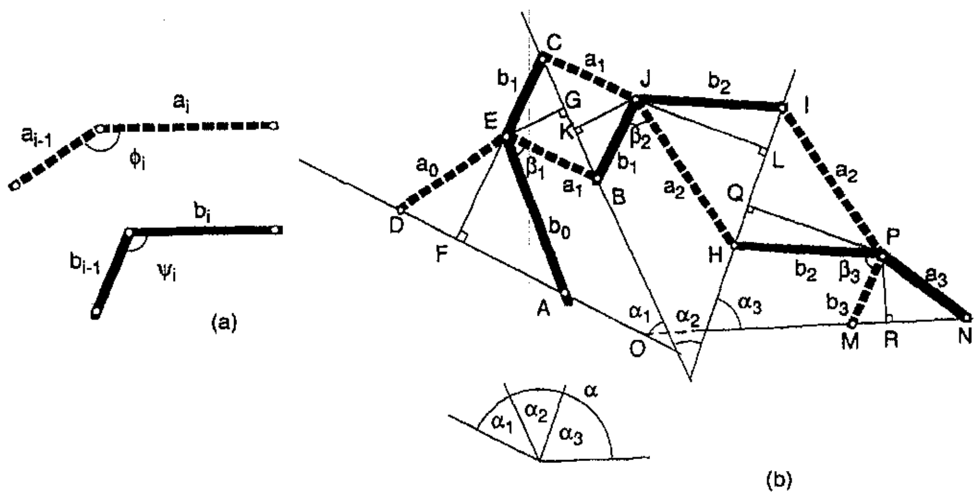


Fig. 7. General Type II GAE consisting of two similar triangles connected by two parallelograms.

(i) each closed loop is a parallelogram
(ii) the triangles on the sides, $\triangle$AED and $\triangle$NPM, are similar, i.e.,

$$\frac{\overline{\mathrm{DE}}}{\overline{\mathrm{PM}}} = \frac{\overline{\mathrm{AE}}}{\overline{\mathrm{NP}}} \quad \text{and} \quad \angle\mathrm{AED} = \angle\mathrm{MPN}. \tag{34}$$

To show that the angle α is constant, we note that eqn (23) is also valid for Type II elements. Also, because of condition (ii)

$$\angle\mathrm{AEF} = \angle\mathrm{NPR} \tag{35}$$

and, hence, eqn (23) is equivalent to

$$\alpha = 3 \times 180° - \Sigma\psi = \text{constant}. \tag{36}$$

It is interesting to notice that, since

$$\angle\mathrm{AED} = \phi_1 - \beta_1 \quad \text{and} \quad \angle\mathrm{MPN} = \psi_3 - \beta_3 \tag{37}$$

and since these angles are equal, eqn (25)—also valid for Type II elements—is equivalent to

$$\Sigma\psi = \Sigma\phi \tag{38}$$

which shows that the sum of the kink angles of the two sets of angulated rods that make up a Type II GAE is constant. For angulated elements consisting of two angulated rods only, eqn (38) becomes $\psi = \phi$, which agrees with eqn (30).

3.3. *An additional property of symmetric GAEs*

The use of foldable elements that embrace a constant angle does not guarantee that a structure made from several elements of this type is (i) foldable, and (ii) maintains its shape during folding. Problems can arise due to a radial shift building up within a GAE, and causing the hinges on one side to move by different amounts to the hinges on the other side. It is also possible for a tangential shift to build up, if the hinges on one side of the element move to a line parallel to the original line. Such problems are particularly critical when designing foldable structures that form *closed loops*.

The easiest way round these difficulties is to use symmetric elements. The motion of a GAE with a single, central axis of symmetry is also symmetric, and hence there will be no radial mismatch between hinges on either side.

If only the angulated elements of Section 2 are considered, i.e., those made from two identical angulated rods, then symmetric configurations are obtained only for elements with

$$\overline{\mathrm{AE}} = \overline{\mathrm{DE}} = \overline{\mathrm{BE}} = \overline{\mathrm{CE}} \quad \text{and} \quad \psi = \phi. \tag{39}$$

If, however, symmetric GAEs are considered, then many different layouts can be obtained. Some examples are given in Section 5.

4. FOLDABLE CIRCULAR STRUCTURES

Figure 8 shows an assembly of eight identical, symmetric, simple angulated elements, each embracing an angle

$$\alpha = \frac{360^\circ}{8} = 45^\circ. \tag{40}$$

These elements form a closed, circular ring structure whose shape can vary continuously between the shapes shown in Fig. 8(a) and Fig. 8(c), through intermediate shapes as shown in Fig. 8(b). In Fig. 8(a) the deployment angle, defined in Section 2, is $\theta = 45^\circ$ and one set

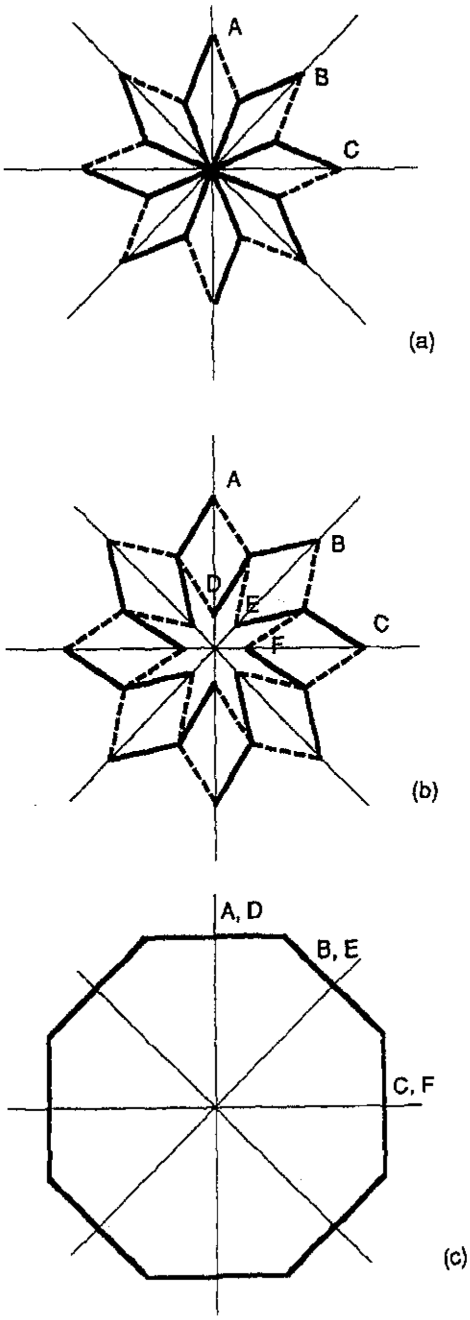


Fig. 8. Foldable ring structure formed by identical angulated rods with a kink angle of 135°. (a) "Expanded" and (c) "retracted" configurations.

of angulated rods (broken lines) is partly hidden by the other set. In Fig. 8(c) the deployment angle is $\theta = 180°$ and one set of rods is completely hidden by the other set.

The hinges of this structure lie on three concentric circles at all times. Note that in Fig. 8(a) one of these circles has shrunk to a point, while in Fig. 8(c) two circles coincide. It is interesting to note that, after reaching a maximum diameter, the circle containing the hinges A, B, C starts to contract, while the other two circles continue to expand. If the physical size of members and joints is neglected, the expansion process terminates when the hinges D, E, and F become coincident with A, B, and C, respectively.

The foldable structure of Fig. 8 is formed by a closed loop of eight *identical rhombuses* and it will be shown in Section 5.1 that in such loops there is no geometric mismatch in different configurations.

The number of angulated elements used in forming a circular loop, as well as the semi-length of the angulated rods, can be varied, but no other changes are possible: all two-dimensional foldable structures with hinges lying on concentric circles are fundamentally of the same type as the structure shown in Fig. 8. This is because each angulated element must be symmetric, and geometric compatibility between adjacent elements requires that all elements remain identical at all stages of folding.

Larger foldable structures based on the solution described above can be formed by inter-connecting two or more concentric circular rings of matching size (Hoberman, 1991). Figure 9 shows the simplest way of doing this, using two *identical ring structures* connected by a series of hinges lying on the circle that contains the hinge A_2. The expansion of this structure is limited by the inner ring becoming fully closed, Fig. 9(a), while its retraction is limited by the outer ring becoming fully stretched. Actually, this is a rather unusual foldable structure. During folding, its outer diameter initially decreases and then increases back to the original value, only the size of the central hole increases monotonically. Actually, this type of behaviour is perfectly suited for large foldable domes, because it is easier to arrange its supports if the perimeter remains approximately constant. The above solution is the key to the Iris Retractable Roof (Hoberman, 1991).

Better packaging of the two concentric rings can be achieved by using a slightly smaller structure for the inner ring. Let $L = \overline{A_2A_3} = \overline{A_3A_4}$ be the semi-length of the angulated elements that make up the outer ring. The optimal value of the semi-length l of the elements of the inner ring, $l = \overline{A_0A_1} = \overline{A_1A_2}$, is such that in the fully-expanded configuration A_0A_2 becomes orthogonal to OP_0. This requires

$$\frac{l}{L} = \cos\frac{\alpha}{2} \tag{41}$$

which, for octagonal rings ($\alpha = 45°$), gives

$$\frac{l}{L} = 0.92.$$

Other values of the ratio l/L are also acceptable, but produce smaller expansion ratios.

Next, it will be shown that *in circular foldable structures made from identical, symmetric angulated elements, contiguous angulated rods can be connected rigidly to one another, to form multi-angulated rods.* Consider two identical angulated rods of semi-length l, lying in neighbouring sectors subtending equal angles α, as shown in Fig. 10. Let node A_2 be the connection point of the two elements. It will be shown that the angle between the two rods, $\angle A_1A_2A_3$, has constant magnitude. Considering the first angulated rod, which lies between the lines OP_0 and OP_2, the distance of hinge A_2 from point O is

$$\overline{OA_2} = \frac{\overline{A_2C_1}}{\sin\alpha/2} = \frac{l\cos(\angle A_1A_2C_1)}{\sin\alpha/2} \tag{42}$$

where

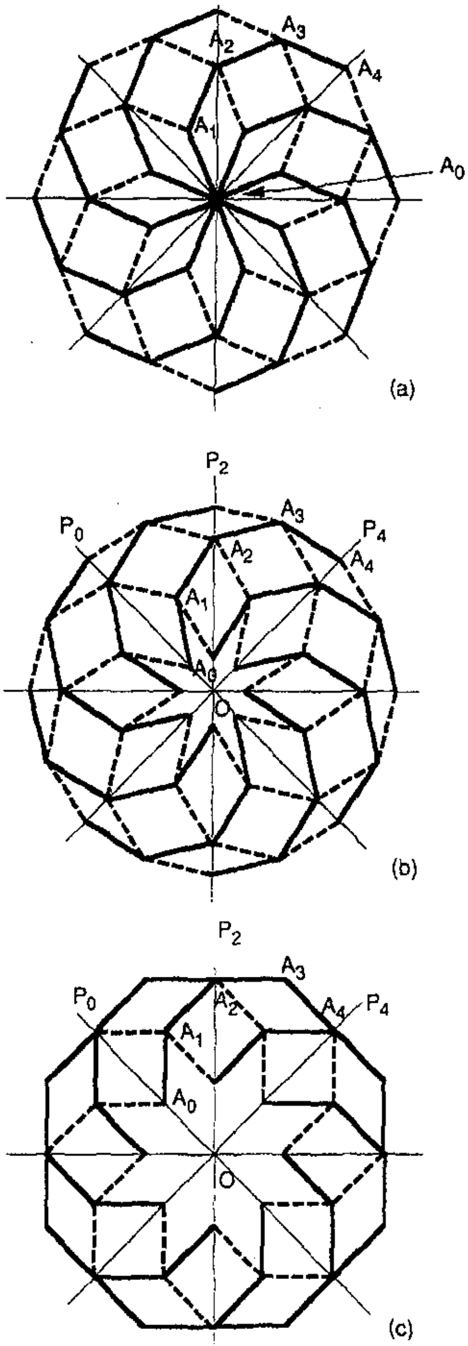


Fig. 9. Foldable circular structure obtained by connecting together two rings like those in Fig. 8. The members $A_0A_1A_2A_3A_4$, etc. are multi-angulated rods.

$$\angle A_1A_2C_1 = 90° - \angle A_2A_1C_1 = 90° - (180° - \angle A_2A_1O). \tag{43}$$

Because

$$\angle A_2A_1O = \angle S_1A_1O + \angle A_2A_1S_1 = \angle S_1A_1O + \frac{180° - \alpha}{2}. \tag{44}$$

Equation (43) becomes

$$\angle A_1A_2C_1 = \angle S_1A_1O - \alpha/2. \tag{45}$$

Substituting eqn (45) into eqn (42)

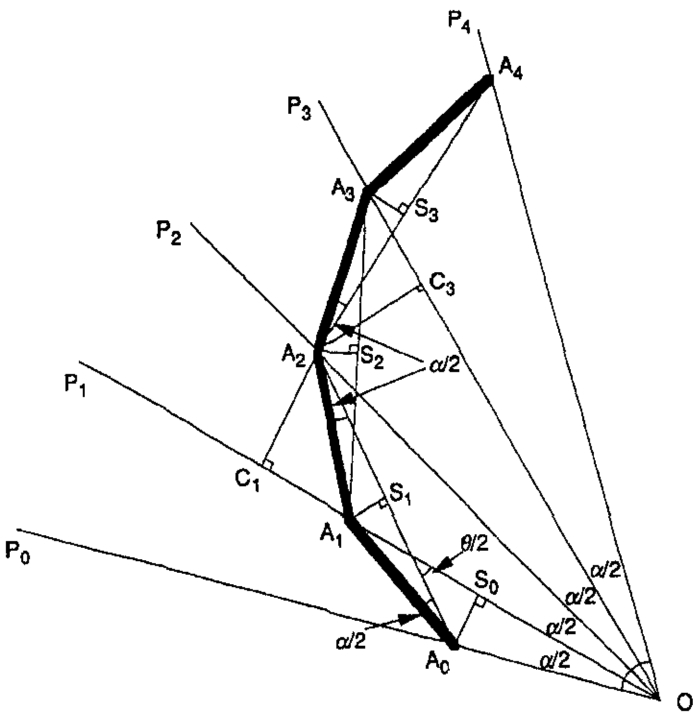


Fig. 10. Multi-angulated rod.

$$\overline{OA_2} = \frac{l\cos(\angle S_1A_1O - \alpha/2)}{\sin\alpha/2}. \tag{46}$$

Also

$$\angle A_1A_2O = 180° - \frac{\alpha}{2} - \angle A_2A_1O = 90° - \angle S_1A_1O. \tag{47}$$

Considering the second angulated rod, the distance of hinge A_2 from point O is

$$\overline{OA_2} = \frac{\overline{A_2C_3}}{\sin\alpha/2} = \frac{l\cos(\angle A_3A_2C_3)}{\sin\alpha/2} \tag{48}$$

where

$$\angle A_3A_2C_3 = \angle S_3A_2C_3 + \alpha/2 = \angle S_3A_3O + \alpha/2. \tag{49}$$

Hence

$$\overline{OA_2} = \frac{l\cos(\angle S_3A_3O + \alpha/2)}{\sin\alpha/2}. \tag{50}$$

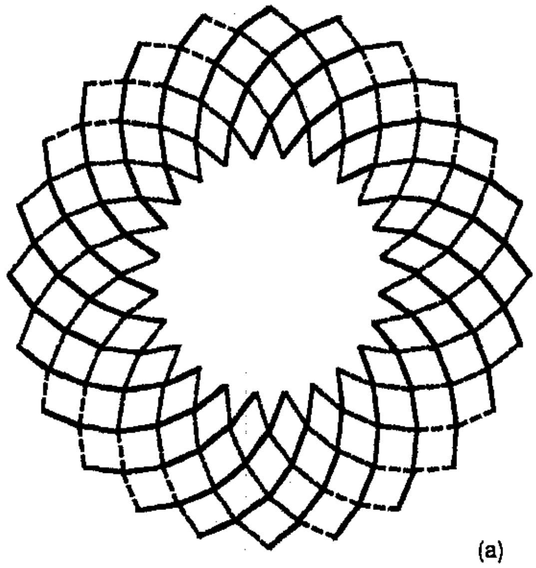
Comparing eqn (46) with eqn (50),

$$\angle S_1A_1O - \alpha/2 = \angle S_3A_3O + \alpha/2. \tag{51}$$

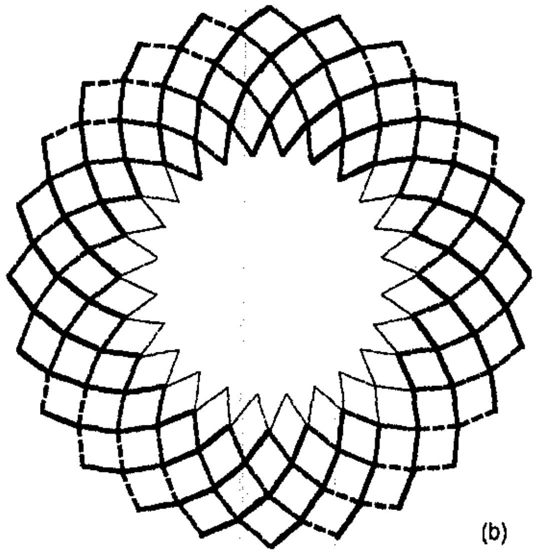
Also

$$\angle A_3A_2O = \angle A_3A_2C_3 + 90° - \alpha/2 = 90° + \angle S_3A_3O. \tag{52}$$

The angle between the two angulated elements can be calculated from eqn (47) and eqns (51)–(52).



(a)



(b)

Fig. 11. Foldable circular structures formed by multi-angulated rods with kink angles of 165°.

$$\begin{aligned}\angle A_1A_2A_3 &= \angle A_1A_2O + \angle A_3A_2O \\ &= 180° + \angle S_3A_3O - \angle S_1A_1O = 180° - \alpha = \text{constant.}\end{aligned} \tag{53}$$

This proof can be extended to any number of contiguous rods of equal semi-length l, provided that they are at a non-decreasing distance from the centre: when they start to turn back towards the centre, the angle $\angle S_iA_iO$ becomes negative and hence the above proof is no longer valid. Subject to this condition, the rods can be rigidly linked together to form a multi-angulated rod with a kink angle of $180° - \alpha$, eqn 53.

Figure 11(a) shows a circular foldable structure containing 48 five-segment multi-angulated rods. This structure has

$$\alpha = \frac{360}{48/2} = 15°.$$

Figure 11(b) shows that modest shape changes can be made by varying the number of segments in some rods. Figure 12 shows photographs of a model structure with

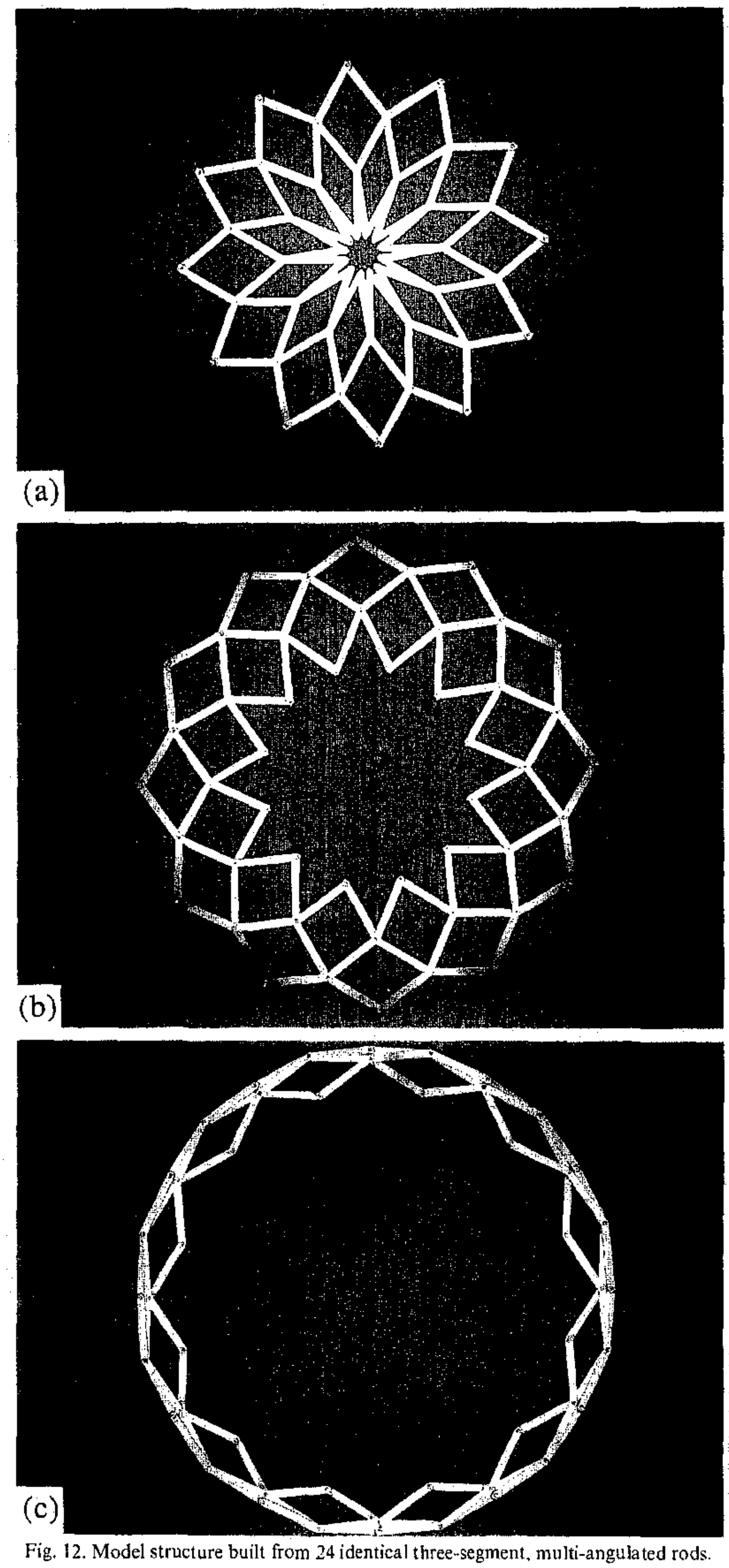


Fig. 12. Model structure built from 24 identical three-segment, multi-angulated rods.

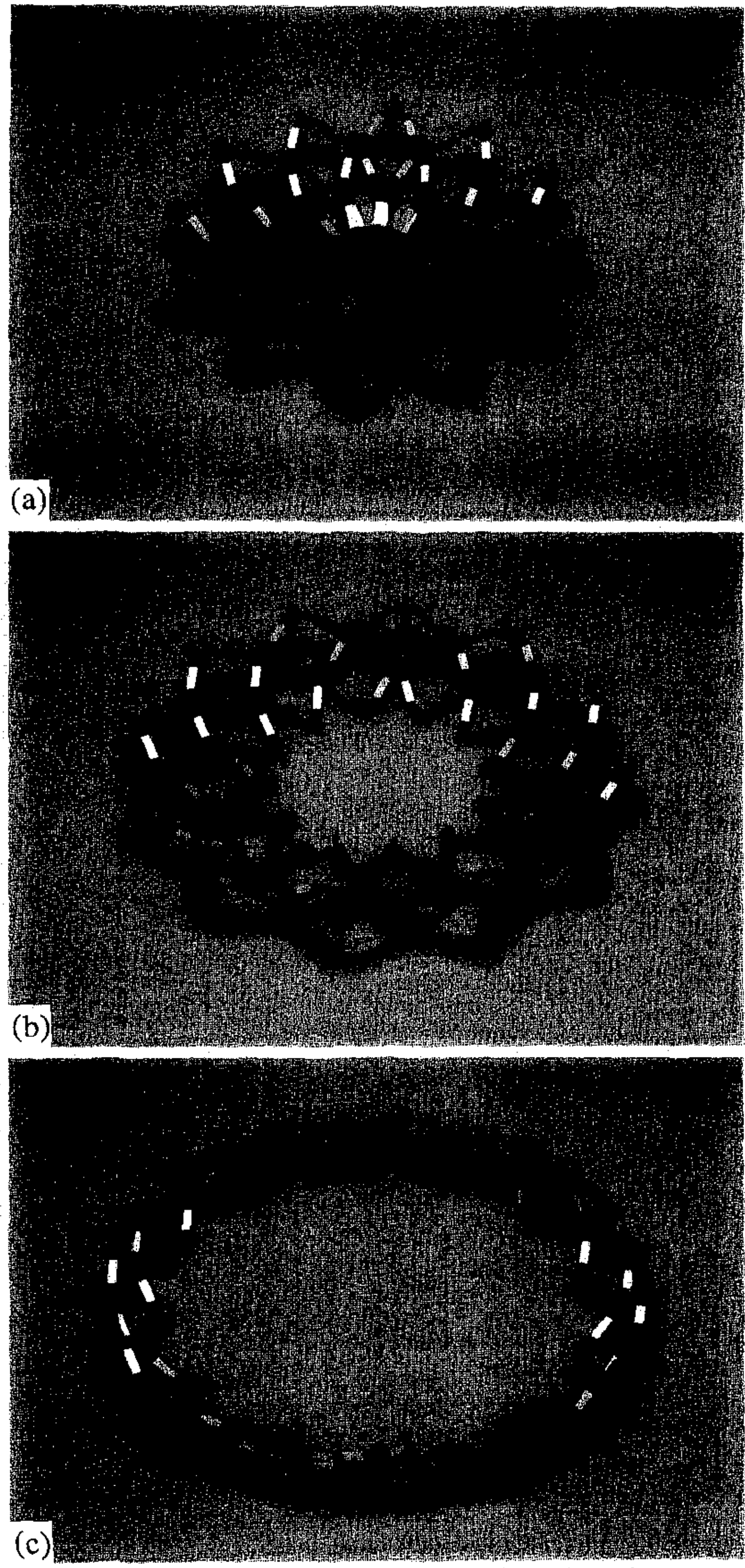


Fig. 19. Double-layer circular foldable structure with curved upper layer.

$$\alpha = \frac{360}{24/2} = 30^\circ$$

whose 24 identical multi-angulated rods have kink angles of 30°, and each rod consists of three segments of length $l = 100$ mm. The fully-deployed and fully-folded configurations of this model are shown in Fig. 12(a) and Fig. 12(c), respectively. Note that the rods cannot fully overlap because of the physical size of the joints.

5. FOLDABLE STRUCTURES OF GENERAL SHAPE

It might be expected that two-dimensional foldable structures with many different shapes might be made by a straightforward extension of the ideas introduced above. Indeed, an obvious way of doing this would be to divide any given boundary shape into straight segments and circular arcs, and then assemble together straight-edged, trellis-type structures of suitable length, Fig. 1, and simple angulated elements with an appropriate angle of embrace. Unfortunately, a bar structure of this type is not foldable. The problem is that, although it is possible to vary the semi-length of the simple angulated elements that make up a circular sector, so that the hinges connecting this sector to its neighbouring trellis-type structure are equally, or proportionally spaced in the radial direction, this can be done only for *a particular configuration*. The scissor hinges do not remain equally spaced when the configuration is varied. Hence, a circular sector cannot be connected to a structure consisting of straight rods, whose scissor hinges are always equally, or proportionally spaced.

To obtain the layout of a two-dimensional foldable structure with a boundary of prescribed shape one must begin by finding a *foldable base structure*, i.e., a structure consisting of angulated rods whose hinges lie on the prescribed boundary. Once a suitable layout for the base structure has been selected, extra members can be connected to it by means of scissor hinges, until the required shape and overall dimensions are obtained. It will be shown that such a structure is foldable and, subject to certain conditions, it remains foldable if contiguous bars are firmly connected, thus forming a series of multi-angulated rods.

5.1. *Layout of the base structure*

Finding a base structure that meets all the shape and folding requirements of a given application is the key to a successful overall solution and yet there is no unique set of rules leading automatically to the best layout of angulated elements. Therefore, the method will be explained by describing the procedure which has been followed for a series of representative examples. All of the examples are of the same basic type, continuous loop structures with a central hole of variable size. Such structures are suitable for foldable roofs for, e.g. sports stadia and tennis courts. Open loop structures are subject to fewer restrictions, and hence much easier to configure using any combination of GAE's.

Figure 13 illustrates a simple technique (Hoberman, 1990) to construct a single-loop foldable bar structure of any shape. Figure 13(a) shows an illustrative, general polygon which may be constructed from a series of Hoberman's elements whose internal hinges coincide with the vertices of the polygon. The semi-length of each angulated rod is equal to half the length of each side and the two rods belonging to the same element form equal kink angles, which are equal to the internal angle of the polygon. Hence, in the fully folded configuration, Fig. 13(b), the elements overlap with the sides of the polygon. Note that half of the angulated rods are hidden by the other rods. In general, of course, these angulated elements are not symmetric and hence, according to Section 3.3, a radial mismatch develops as the structure is folded. However, the overall mismatch adds up to zero, Fig. 13(c), because in this case the angulated elements form a chain of *similar rhombuses* whose diagonals are reduced in length by proportional amounts and also remain at constant angles during folding.

Figure 14 shows a more general type of closed loop structure, whose internal hinges also coincide with the vertices of the polygon of Fig. 13(a). Here, the angulated rods making

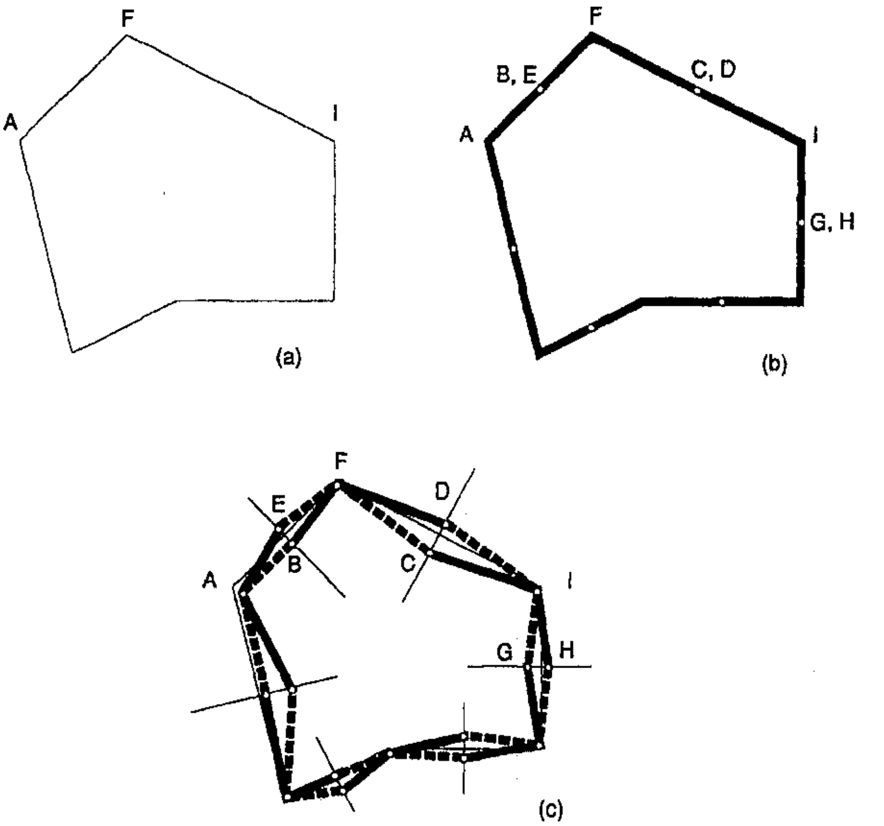


Fig. 13. Foldable closed loop structure which folds along a given polygon. The angulated elements form similar rhombuses.

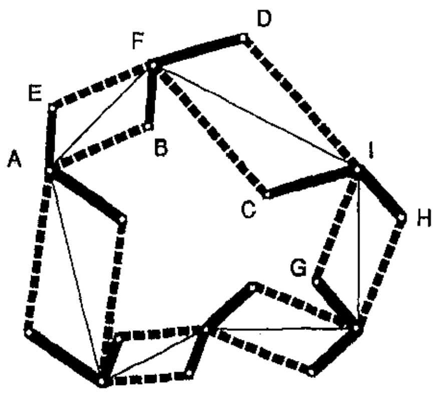


Fig. 14. Foldable closed loop structure which folds along the polygon of Fig. 13, but consists of similar parallelograms.

up each element are no longer identical, but still have a kink angle equal to the internal angle of the polygon, and form a chain of *similar parallelograms*. This property implies that the loop structure is foldable, because the sides of the polygon vary by proportional amounts and hence no geometric mismatch builds up during folding. Note that each of the six angulated elements used in this solution is a Type II GAE without any parallelograms, as that shown in Fig. 6.

In addition to the above solutions for base structures forming closed loops of any shape, greater freedom is available in the case of loops with one or more axes of symmetry. Basically, any GAE can be used to form the basic repeating unit and since, by symmetry, all units behave in the same way, geometric compatibility in all configurations is automatically

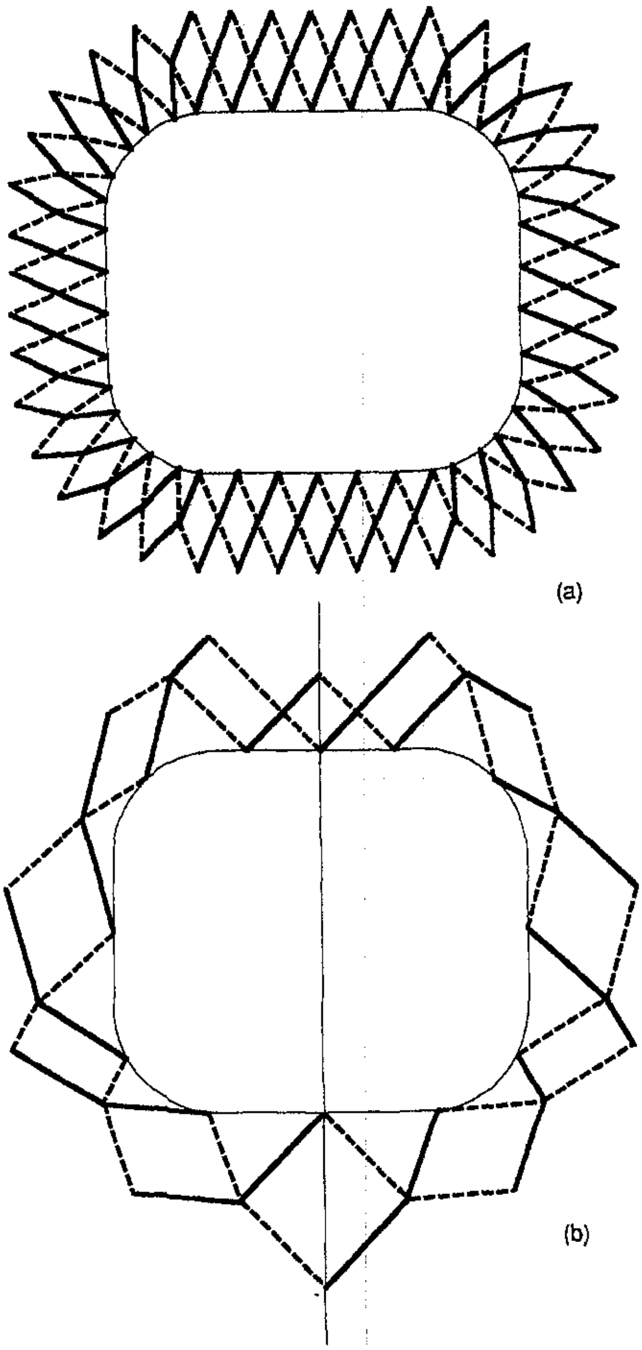


Fig. 15. Closed loop foldable structures whose internal boundary has the shape of a rectangle with rounded corners. Layout (a) consists of identical rhombuses, while (b) is based on a symmetric GAE of both Type I and Type II.

satisfied. Figure 15 shows two loop structures whose innermost hinges lie on a rectangle with rounded corners. The base structure shown in Fig. 15(a) consists of identical rhombuses, and hence there is no need to invoke symmetry to prove that this structure is foldable. The base structure shown in Fig. 15(b), though, is based on a symmetric arrangement of GAEs which are both of Type I and Type II. This can be seen by means of the central line of symmetry that has been drawn in Fig. 15(b), which divides two opposite rhombuses into pairs of similar isosceles triangles.

5.2. *How to extend the base structure*

Any base structure can be extended by the addition of a pair of bars of any length, connected to one another and to the base structure by scissor hinges. The resulting structure

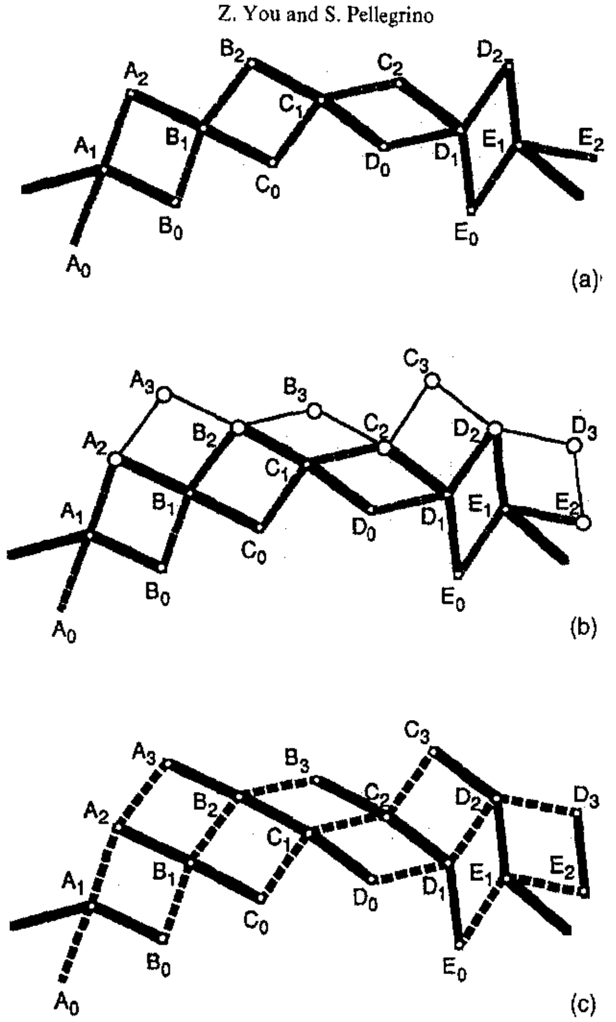


Fig. 16. Three foldable structures: (a) base structure consisting of several parallelograms; (b) extended structure, with additional hinged bars; (c) rigidly-connected structure formed by multi-angulated rods.

will be foldable, like the original base structure, provided that the members added to it are not collinear. Repeating the same argument it can be shown that any number of pairs of bars connected by hinges to the base structure will leave its mobility unchanged.

Of greatest practical importance is the particular case of a base structure consisting of a series of parallelograms, as in Figs 13–15.

Figure 16(a) shows a general, small part of a bar structure consisting of angulated elements. Additional members are connected to its outer hinges, Fig. 16(b), such that the quadrangles $A_2A_3B_1B_2$, etc. are *parallelograms*. This extended structure is foldable because all additional members are free to rotate with respect to the base structure but, in fact, no relative rotation between consecutive rods occurs as the structure is folded. i.e., $\angle A_1A_2A_3$, $\angle B_1B_2B_3$, etc. remain constant. Consider, for example, $\angle A_1A_2A_3$. Because A_1A_2 and A_2A_3 remain parallel to B_0B_1 and B_1B_2, respectively,

$$\angle A_1A_2A_3 = \angle B_0B_1B_2 = \text{constant} \tag{54}$$

since $\angle B_0B_1B_2$ is the kink angle of an angulated rod, which is fixed.

In conclusion, this foldable structure can be made from multi-angulated rods similar to those introduced in Section 4, as shown in Fig. 16(c), but note that the kink angles along these rods are no longer equal. The same procedure is valid for all other closed loop base

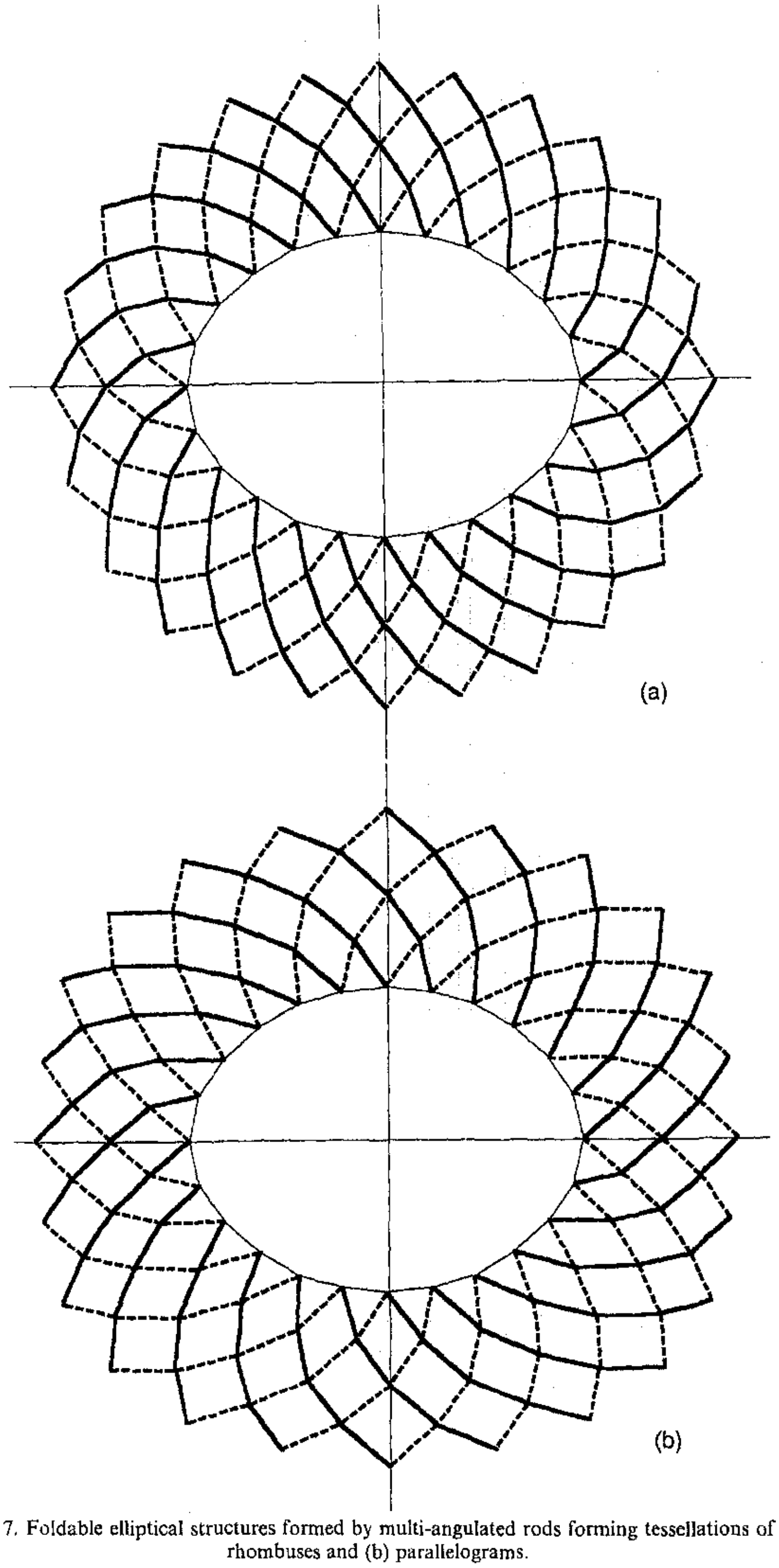
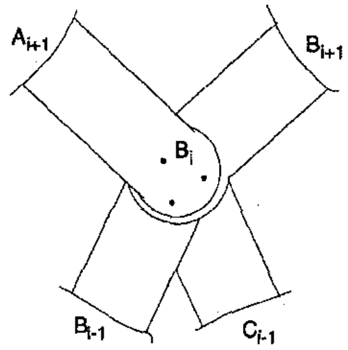
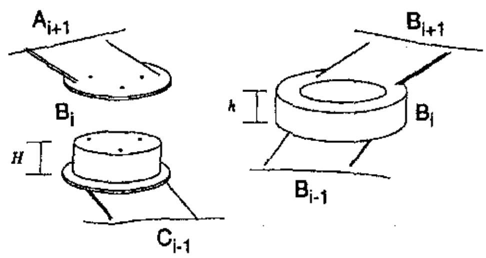


Fig. 17. Foldable elliptical structures formed by multi-angulated rods forming tessellations of (a) rhombuses and (b) parallelograms.

structures discussed in this section, as for any open loop base structure. Figure 17 shows two symmetric foldable structures whose internal boundaries have an identical elliptical shape. In both structures, the inner joints are equally spaced but, while the first layout is a tessellation of rhombuses whose side lengths are all equal, the layout of Fig. 17(b) is a tessellation of parallelograms, whose side lengths are not all equal.



(a)



(b)

Fig. 18. Detail of joint of dome structure.

6. DOME STRUCTURES

The two-dimensional solutions derived above easily extend to curved structures, by projecting any two-dimensional solution onto a surface with the required shape. Thus, each multi-angulated rod can be curved out of plane. Of course, all connectors between multi-angulated rods should be perpendicular to the plane of projection.

The folding angle may be restricted if the rods are not allowed to overlap during folding. This problem can be solved by a proper design of the connections. An example of a suitable connector between two rods is shown in Fig. 18. In the drawings, $B_{i-1}\ B_i\ B_{i+1}$ is one of the rods, whilst the other rod $A_{i+1}\ B_i\ C_{i-1}$ is in two parts. $B_i\ C_{i-1}$ has a circular cross-section cylindrical post of height H and $A_{i+1}\ B_i$ has a cap which can be fitted onto the post. $B_{i-1}\ B_i\ B_{i+1}$ is fitted with an open ring at B_i of height $h < H$, so that a relative rotation of the two rods can take place.

Figure 19 shows a double layer model structure whose curved top layer is connected to the flat bottom layer by long bolts. The bottom layer is identical to the model shown in Fig. 12, and the orthogonal projection of the top layer onto the plane of the bottom layer is also identical to it. This model folds until the outer rods overlap fully, and thus demonstrates that the interference between rods connected to the same hinge has been successfully eliminated. Note that bracing elements could be added between the upper and lower cords, to increase the stiffness of the structure, if desired.

7. DISCUSSION AND CONCLUSIONS

A general method for the design of two-dimensional foldable structures has been introduced. The new method extends and generalises the trellis-type structures, based on a tiling of parallelograms whose sides are collinear, to structures based on any tiling of parallelograms. It has been shown that a bar structure of this type is (i) foldable and (ii) can be made from multi-angulated, rigid rods connected by scissor hinges. This result affords much greater freedom in the range of shapes that can be achieved, and of boundary conditions that can be met. This approach can be easily extended to three-dimensional dome structures.

Also, a family of elements for foldable structures has been introduced. These elements consist of angulated rods connected by scissor hinges. It has been shown that any element bounded by either isosceles triangles or similar triangles, with any number of parallelograms in between, maintains a constant angle of embrace.

Finally, a method for the design of structures consisting of multi-angulated rods that fold along their perimeter has been introduced, and there is practically no limit to the range of perimeter shapes that can be achieved.

These new solutions have significant implications for the design of foldable structures. Because continuous members are used throughout, the complexity of the joint is reduced and more efficient structural design become possible, e.g., curved trusses can be used instead of beams. Also, the attachment of covering panels or membranes is simplified.

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APPENDIX B

Section properties of the selected faces of hoberman's angulated element			
Area = 16175.86 millimeters^2			
Centroid relative to assembly origin: (millimeters)			
	X = -86.57		
	Y = -174.58		
	Z = -9.90		
Moments of inertia of the area, at the centroid: (millimeters ^ 4)			
	Lxx = 38679056.53	Lxy = -402631.76	Lxz = -10939361.94
	Lyx = -402631.76	Lyy = 95310772.16	Lyz = -1668133.46
	Lzx = -10939361.94	Lzy = -1668133.46	Lzz = 129560809.49
Polar moment of inertia of the area, at the centroid = 130931718.72 millimeters ^ 4			
Principal moments of inertia of the area, at the centroid: (millimeters ^ 4)			
	Ix = 37374696.59		
	Iy = 95244222.88		
Moments of inertia of the area, at the output coordinate system: (millimeters ^ 4)			
	LXX = 159902915.32	LXY = 244071592.26	LXZ = 2920448.45
	LYX = 244071592.26	LYY = 588346110.32	LYZ = 26283182.60
	LZX = 2920448.45	LZY = 26283182.60	LZZ = 131145434.38
Mass properties of selected components			
Output coordinate System: -- default --			
The center of mass and the moments of inertia are output in the coordinate system of walltrialhoberman1			
Mass = 20.24 grams			
Volume = 20237.50 cubic millimeters			
Surface area = 12613.46 millimeters^2			
Center of mass: (millimeters)			
	X = -157.25		
	Y = -186.25		
	Z = -25.17		
Principal axes of inertia and principal moments of inertia: (grams * square millimeters)			
Taken at the center of mass.			

Ix = (0.99, -0.14, 0.03) Px = 33754.61
Iy = (0.14, 0.99, 0.10) Py = 53846.60
Iz = (-0.04, -0.09, 0.99) Pz = 87180.75
Moments of inertia: (grams * square millimeters)
Taken at the center of mass and aligned with the output coordinate system.
Lxx = 34236.85 Lxy = -2940.00 Lxz = 2051.03
Lyx = -2940.00 Lyy = 53729.16 Lyz = 2968.10
Lzx = 2051.03 Lzy = 2968.10 Lzz = 86815.95
Moments of inertia: (grams * square millimeters)
Taken at the output coordinate system.
Ixx = 749103.54 Ixy = 589795.38 Ixz = 82143.66
Iyx = 589795.38 Iyy = 566990.51 Iyz = 97831.52
Izx = 82143.66 Izy = 97831.52 Izz = 1289307.38

alpha	total	phi	psi
60	240	130	110
60	240	130	110
60	240	130	110
180	720	390	330

Mass properties of part1 Type I GAE (Part Configuration - Default)		
Output coordinate System: -- default --		
Density = 0.00 grams per cubic millimeter		
Mass = 10.63 grams		
Volume = 10634.05 cubic millimeters		
Surface area = 6617.79 millimeters^2		
Center of mass: (millimeters)		
X = -4.55		
Y = -21.48		
Z = 2.50		
Principal axes of inertia and principal moments of inertia: (grams * square millimeters)		
Taken at the center of mass.		
Ix = (0.86, 0.51, -0.00)	Px = 1792.60	
Iy = (-0.51, 0.86, -0.00)	Py = 33750.67	
Iz = (-0.00, 0.00, 1.00)	Pz = 35499.04	
Moments of inertia: (grams * square millimeters)		
Taken at the center of mass and aligned with the output coordinate system.		
Lxx = 10111.14	Lxy = 14023.06	Lxz = -3.19
Lyx = 14023.06	Lyy = 25432.14	Lyz = -2.76
Lzx = -3.19	Lzy = -2.76	Lzz = 35499.04
Moments of inertia: (grams * square millimeters)		
Taken at the output coordinate system.		
Ixx = 15084.20	Ixy = 15063.07	Ixz = -124.43
Iyx = 15063.07	Iyy = 25719.27	Iyz = -574.75
Izx = -124.43	Izy = -574.75	Izz = 40625.87

Mass properties of part2 Type I GAE (Part Configuration - Default)
Output coordinate System: -- default --
Density = 0.00 grams per cubic millimeter
Mass = 10.70 grams
Volume = 10702.02 cubic millimeters
Surface area = 6699.03 millimeters^2
Center of mass: (millimeters)
X = 4.52
Y = -28.43
Z = 2.49
Principal axes of inertia and principal moments of inertia: (grams * square millimeters)
Taken at the center of mass.
Ix = (0.72, 0.70, -0.00) Px = 3211.17
Iy = (-0.70, 0.72, -0.00) Py = 29106.06
Iz = (-0.00, 0.00, 1.00) Pz = 32272.71
Moments of inertia: (grams * square millimeters)
Taken at the center of mass and aligned with the output coordinate system.
Lxx = 15825.73 Lxy = 12943.17 Lxz = 0.85
Lyx = 12943.17 Lyy = 16491.50 Lyz = -3.26
Lzx = 0.85 Lzy = -3.26 Lzz = 32272.71
Moments of inertia: (grams * square millimeters)
Taken at the output coordinate system.
Ixx = 24545.13 Ixy = 11568.39 Ixz = 121.28
Iyx = 11568.39 Iyy = 16776.33 Iyz = -761.26
Izx = 121.28 Izy = -761.26 Izz = 41144.13

Mass properties of Part1 Type II GAE (Part Configuration - Default)
Output coordinate System: -- default --
Density = 0.00 grams per cubic millimeter
Mass = 10.69 grams
Volume = 10690.10 cubic millimeters
Surface area = 6684.33 millimeters^2
Center of mass: (millimeters)
X = -0.46
Y = -25.32
Z = 2.50
Principal axes of inertia and principal moments of inertia: (grams * square millimeters)
Taken at the center of mass.
Ix = (0.79, 0.61, 0.00) Px = 2487.60
Iy = (-0.61, 0.79, 0.00) Py = 31539.36
Iz = (0.00, 0.00, 1.00) Pz = 33982.42
Moments of inertia: (grams * square millimeters)
Taken at the center of mass and aligned with the output coordinate system.
Lxx = 13205.36 Lxy = 14017.82 Lxz = 0.00
Lyx = 14017.82 Lyy = 20821.61 Lyz = 0.00
Lzx = 0.00 Lzy = 0.00 Lzz = 33982.42
Moments of inertia: (grams * square millimeters)
Taken at the output coordinate system.
Ixx = 20123.77 Ixy = 14143.02 Ixz = -12.36
Iyx = 14143.02 Iyy = 20890.71 Iyz = -676.59
Izx = -12.36Izy = -676.59 Izz = 40836.31

Mass properties of Part2 Type II GAE (Part Configuration - Default)
Output coordinate System: -- default --
Density = 0.00 grams per cubic millimeter
Mass = 8.04 grams
Volume = 8035.84 cubic millimeters
Surface area = 5111.92 millimeters^2
Center of mass: (millimeters)
X = -0.82
Y = -17.69
Z = 2.49
Principal axes of inertia and principal moments of inertia: (grams * square millimeters)
Taken at the center of mass.
Ix = (0.80, 0.60, 0.00) Px = 1102.91
Iy = (-0.60, 0.80, -0.00) Py = 13487.85
Iz = (-0.00, 0.00, 1.00) Pz = 14557.43
Moments of inertia: (grams * square millimeters)
Taken at the center of mass and aligned with the output coordinate system.
Lxx = 5630.70 Lxy = 5964.52 Lxz = 2.18
Lyx = 5964.52 Lyy = 8960.07 Lyz = 1.42
Lzx = 2.18 Lzy = 1.42 Lzz = 14557.43
Moments of inertia: (grams * square millimeters)
Taken at the output coordinate system.
Ixx = 8195.43 Ixy = 6080.68 Ixz = -14.16
Iyx = 6080.68 Iyy = 9015.22 Iyz = -352.43
Izx = -14.16 Izy = -352.43 Izz = 17077.74