UNIVERSITI TEKNOLOGI PETRONAS

FINAL EXAMINATION
MAY 2015 SEMESTER

COURSE	**:**	**ZAB1043 – MATHEMATICAL METHODS FOR PHYSICS**
DATE	**:**	**3rd SEPTEMBER 2015 (THURSDAY)**
TIME	**:**	**2.30 PM – 5.30 PM (3 hours)**

INSTRUCTIONS TO CANDIDATES

1. Answer **ALL** questions from this Question Booklet.
2. Begin **EACH** answer on a new page in the Answer Booklet given.
3. Indicate clearly answers that are cancelled, if any.
4. Where applicable, clearly indicate steps taken in arriving at the solutions and state **ALL** assumptions, if any.
5. Do not open this Question Booklet until instructed.

Note : i. There are **THIRTEEN (13)** printed pages including the cover page and Appendices.

ii. Engineering Data & Formulae Booklet will be provided.

1. a. Given f is a scalar function and $\vec{V} = \langle V_1, V_2, V_3 \rangle$ is a vector. Show that $\vec{\nabla} \times (f\vec{V}) = f\,\vec{\nabla} \times \vec{V} + (\vec{\nabla} f) \times \vec{V}$.

[8 marks]

b. In the Pauli theory of the electron one encounters the expression

$$(\vec{p} - e\vec{A}) \times (\vec{p} - e\vec{A})\Phi$$

where Φ is a scalar function and $\vec{A}$ is the magnetic vector potential related to the magnetic induction $\vec{B}$ by $\vec{B} = \vec{\nabla} \times \vec{A}$. Given that $\vec{p} = -i\vec{\nabla}$ where i is the imaginary unit that satisfies $i^2 = -1$. By using the result in **part (a)**, show that this expression reduces to $ie\vec{B}\Phi$.

[6 marks]

c. Given that the successive operations of the vector differential operator to a vector function $\vec{A}$ possesses the identity $\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla}(\vec{\nabla} \bullet \vec{A}) - (\vec{\nabla} \bullet \vec{\nabla})\vec{A}$. Show that if the vector function $\vec{A}$ satisfies the solenoidal condition, $\vec{\nabla} \bullet \vec{A} = 0$, any solution of the equation

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) - k^2 \vec{A} = 0$$

automatically satisfies the vector Helmholtz equation

$$(\vec{\nabla}^2 + k^2)\vec{A} = 0$$

where $\vec{\nabla}^2 = \vec{\nabla} \bullet \vec{\nabla}$.

[6 marks]

2. a. By using the divergence theorem, compute the outward flux

$$\oiint_S \vec{F}(x, y, z) \bullet d\vec{S}$$

of the vector field

$$\vec{F}(x, y, z) = \left\langle \ln\left(y^2 + \frac{1}{z}\right), 2y - \exp\left(\frac{\sin x}{z}\right), \frac{1}{x^2 - y^2} + z \right\rangle$$

over the closed surface S of a sphere with radius 3 units centered at the origin.

[5 marks]

b. By using Stokes Theorem, evaluate the line integral

$$\oint_C \vec{F} \bullet d\vec{r}$$

where $\vec{F}(x, y, z) = \langle 3z, 4x, 2y \rangle$ and C is the boundary of the paraboloid $z = 4 - x^2 - y^2$ for which $z \geq 0$ with upward orientation.

[8 marks]

c. Compute the work done by the force, $\vec{F}(x, y, z) = xy\hat{i} + 3z\hat{j} + \hat{k}$, along the curve $x = \cos t, y = \sin t$ and $z = 2t$ from the point $(1, 0, 0)$ to $(0, 1, \pi)$.

[7 marks]

3. a. Given a matrix $A=\begin{pmatrix}1 & 2\\ 2 & 4\end{pmatrix}$

i. Find the eigenvalues of matrix A.

[2 marks]

ii. Find the eigenvectors for all the eigenvalues found in part (a)(i).

[4 marks]

iii. By diagonalizing the matrix A, compute A^5.

[4 marks]

b. i. Let $F(x,y,y')$ be a function having continuous second order partial derivatives. Show that if the functional

$$I[y]=\int_a^b F(x,y(x),y'(x))dx\,,$$

$y(a)=\alpha,\ y(b)=\beta,\ y(x)\in C^2[a,b]$, has a local extremum $y_0(x)$, then $y_0(x)$ satisfies Euler's equation

$$\frac{\partial F}{\partial y}-\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)=0\,.$$

[7 marks]

ii. If $F=1+(y')^2,\ y_0(0)=0,\ y_0(1)=1$, find the extremal.

[3 marks]

4. Suppose $z = x + iy$ is a complex number.

a. Let $v(x, y) = 2xy$,

i. by using Laplace equation show that $v(x, y)$ is harmonic.

[3 marks]

ii. find the entire function $f(z) = u(x, y) + iv(x, y)$.

[4 marks]

b. i. Show whether $f(z) = e^{\frac{1}{2}z}$ is analytic or not.

(Hint: $e^{\pm iy} = \cos y \pm i \sin y$)

[4 marks]

ii. Evaluate

$$\int_{8+\pi i}^{8-3\pi i} e^{\frac{1}{2}z}\, dz$$

[4 marks]

c. By using Residue Integration method, evaluate

$$\oint_C \frac{e^{\frac{1}{2}z}}{z^2+1}\, dz$$

where C is the circle $|z + i| = 1$ oriented counterclockwise.

[5 marks]

5. In order to answer the following question, you need to refer to **APPENDIX II** (Sturm-Liouville eigenvalue/eigenfunction problem).

a. Given the eigenfunction

$$\psi_n(x) = H_n(\alpha\, x)\, e^{\frac{-\alpha^2 x^2}{2}}$$

where α is a constant and $H_n(\alpha\, x)$ is the Hermite polynomials.

By making use of the differential equation for the Hermite polynomials, show that

$$\psi_n''(x) = -\alpha^2(2n+1-\alpha^2 x^2)\psi_n(x)$$

[8 marks]

b. If the eigenfunction $\psi_n(x)$ in **part (a)** satisfies the Harmonic oscillator time independent Schrödinger equation

$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{k}{2}x^2\right)\psi_n(x) = E_n\,\psi_n(x) \qquad (1)$$

where $\hbar$, m and k are constants, and the natural frequency of the harmonic oscillator is given by $\omega = \sqrt{\frac{k}{m}}$, express the constant α in terms of $\hbar$, m and ω.

[8 marks]

c. By using the result in **part (b)**, show that the eigenvalue of equation (1) is given by

$$E_n = \hbar\omega\left(n + \tfrac{1}{2}\right).$$

[4 marks]

~ END OF PAPER ~

APPENDIX I

1. Cauchy-Riemann Equations

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y},\ \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x} \quad \text{where } f(z)=u(x,y)+iv(x,y)$$

2. Laplace Equations

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0\ ,\quad \frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=0$$

3. Cauchy's Integral Formula

$$\oint_C \frac{f(z)}{z-z_0}\,dz=2\pi i\, f(z_0)$$

4. curl $\vec{F}$, $\vec{F}=P\hat{i}+Q\hat{j}+R\hat{k}$

$$\nabla\times\vec{F}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ P & Q & R\end{vmatrix}$$

5. Divergence $\vec{F}$, $\vec{F}=P\hat{i}+Q\hat{j}+R\hat{k}$

$$\text{div}\,\vec{F}=\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}$$

6. Divergence Theorem

$$\oiint_S \vec{F}\bullet d\vec{S}=\iiint_E \text{div}\,\vec{F}\,dV$$

7. Stoke's Theorem

$$\oint_C \vec{F}\bullet d\vec{r}=\iint_S \text{curl}\,\vec{F}\bullet n\,dS=\iint_S(\nabla\times\vec{F})\bullet\vec{n}\,ds$$

8. If $f(z)$ has simple pole at $z=z_0$,

$$\operatorname*{Res}_{z=z_0} f(z)=\lim_{z\to z_0}(z-z_0)f(z)$$

9. If $f(z)=\frac{p(z)}{q(z)}$ has simple pole at $z=z_0$,

$$\operatorname*{Res}_{z=z_0} f(z)=\operatorname*{Res}_{z=z_0}\frac{p(z)}{q(z)}=\frac{p(z_0)}{q'(z_0)}$$

10. If $f(z)$ has m^{th}-order pole at $z = z_0$,

$$\operatorname*{Res}_{z=z_0} f(z) = \frac{1}{(m-1)!} \lim_{z \to z_0} \left\{ \frac{d^{m-1}}{dz^{m-1}} \left[(z - z_0)^m f(z) \right] \right\}$$

11. Residue Integration Method

$\oint_C f(z)dz = 2\pi i \sum_{i=0}^{n} \operatorname*{Res}_{z=z_i} f(z)$ for the poles $z_0, z_1, \ldots, z_n$ lying inside the contour C.

12. Diagonalization of matrix A, $P^{-1}AP = D$ where $P = \begin{pmatrix} v_1 & v_2 & \cdots & v_n \end{pmatrix}$ and $D = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}$ where v_i is the eigenvector corresponding to the eigenvalue λ_i.

13. Taylor's series expansion about $x = x_0$

$$f(x) = f(x_0) + (x - x_0) f'(x_0) + \frac{1}{2!}(x - x_0)^2 f''(x_0) + \cdots$$

Let $\Delta x = x - x_0$,

$$f(x_0 + \Delta x) = f(x_0) + \Delta x\, f'(x_0) + \frac{1}{2!}(\Delta x)^2 f''(x_0) + \cdots$$

APPENDIX II

Sturm-Liouville eigenvalue/eigenfunction problem

The problem is to solve the differential equation

$$\frac{d}{dx}\left(f(x)\frac{dy_n(x)}{dx}\right)-g(x)y_n(x)+\lambda_n w(x)y_n(x)=0$$

for all possible eigenvalues λ_n and eigenfunctions $y_n(x)$ where $f(x)$, $g(x)$ and $w(x)$ are all assumed to be real with $w(x)\geq 0$ on the range $a\leq x\leq b$. The pair of eigenfunctions $y_m(x)$ and $y_n(x)$ must satisfy the boundary condition:

$$y_m^*(a)y_n'(a)f(a)=y_m^*(b)y_n'(b)f(b)$$

Each function has a normalization factor defined by

$$h_n=\int_a^b w(x)|y_n(x)|^2\,dx$$

Useful integrals:

$$\int_0^\infty x^n e^{-x}dx=n! \qquad \text{for} \qquad n=0,1,2,3,\ldots$$

$$\int_{-\infty}^\infty x^{2n}e^{-x^2}dx=\frac{\sqrt{\pi}}{2^n}(2n-1)!! \qquad \text{for} \qquad n=0,1,2,3,\ldots$$

where $(-1)!!=1$, $1!!=1$, $3!!=3$, $5!!=3.5$, $7!!=3.5.7$, etc.

I) Fourier Series, $\phi_n(x)$, $n=0,\pm1,\pm2,\ldots$

Range:

$$x\in[a,b]$$

Functions of Sturm-Liouville equation:

$$w(x)=1$$
$$f(x)=1$$
$$g(x)=0$$

Differential equation: $\phi_n''(x)+\lambda_n\phi_n(x)=0$

Eigenvalue: $\lambda_n=(nk)^2$

Eigenfunction: $\phi_n(x) = e^{inkx}$

Periodic boundary condition:
$$\phi_n(a) = \phi_n(b)$$
$$\phi_n'(a) = \phi_n'(b)$$

II) Hermite Polynomials, $H_n(x)$, $n = 0, 1, 2, \ldots$

Range:

$$x \in (-\infty, \infty)$$

Functions of Sturm-Liouville equation:

$$w(x) = e^{-x^2}$$
$$f(x) = e^{-x^2}$$
$$g(x) = 0$$

Differential equation:

$$H_n''(x) - 2xH_n'(x) + \lambda_n H_n(x) = 0$$

Eigenvalue:

$$\lambda_n = 2n$$

Eigenfunction:

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}$$

Ladder operations:

$$H_{n+1}(x) = 2xH_n(x) - H_n'(x)$$
$$H_{n-1}(x) = \frac{H_n'(x)}{2n}$$

Explicit expressions:

$$H_0(x) = 1$$
$$H_1(x) = 2x$$
$$H_2(x) = 4x^2 - 2$$
$$H_3(x) = 8x^3 - 12x$$

III) Laguerre Polynomials, $L_n(x)$, $n = 0, 1, 2, \ldots$

Range:

$$x \in [0, \infty)$$

Functions of Sturm-Liouville equation:

$$w(x) = e^{-x}$$
$$f(x) = xe^{-x}$$
$$g(x) = 0$$

Differential equation:

$$xL_n''(x) + (1-x)L_n'(x) + \lambda_n L_n(x) = 0$$

Eigenvalue:

$$\lambda_n = n$$

Eigenfunction:

$$L_n(x) = \frac{e^x}{n!}\frac{d^n}{dx^n}\left(x^n e^{-x}\right)$$

Ladder operations:

$$L_{n+1}(x) = \frac{(n+1-x)L_n(x) + xL_n'(x)}{n+1}$$
$$L_{n-1}(x) = \frac{L_n(x) - xL_n'(x)}{n}$$

Explicit expressions:

$$L_0(x) = 1$$
$$L_1(x) = -x+1$$
$$L_2(x) = \frac{x^2-4x+2}{2}$$
$$L_3(x) = \frac{-x^3+9x^2-18x+6}{6}$$

IV) Lagendre Polynomials, $P_n(x)$, $n = 0, 1, 2, \ldots$

Range:

$$x \in [-1, 1]$$

Functions of Sturm-Liouville equation:

$$w(x) = 1$$
$$f(x) = 1-x^2$$
$$g(x) = 0$$

Differential equation:

$$(1-x^2)P_n''(x) - 2xP_n'(x) + \lambda_n P_n(x) = 0$$

Eigenvalue:

$$\lambda_n = n(n+1)$$

Eigenfunction:

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$$

Ladder operations:

$$P_{n+1}(x) = \frac{xP_n(x)-(1-x^2)P_n'(x)}{n+1}$$

$$P_{n-1}(x) = \frac{xP_n(x)+(1-x^2)P_n'(x)}{n}$$

Explicit expressions:

$$P_0(x) = 1$$
$$P_1(x) = x$$
$$P_2(x) = \frac{3x^2-1}{2}$$
$$P_3(x) = \frac{5x^3-3x}{2}$$

V) Associated Laguerre Polynomials, $L_n^m(x)$, $n, m = 0, 1, 2, \ldots$

Range:

$$x \in [0, \infty)$$

Functions of Sturm-Liouville equation:

$$w(x) = x^m e^{-x}$$
$$f(x) = x^{m+1} e^{-x}$$
$$g(x) = 0$$

Differential equation:

$$xL_n^m{}''(x) + (m+1-x)L_n^m{}'(x) + \lambda_n L_n^m(x) = 0$$

Eigenvalue:

$$\lambda_n = n$$

Eigenfunction:

$$L_n^m(x) = \frac{e^x x^{-m}}{n!}\frac{d^n}{dx^n}\left(x^{n+m}e^{-x}\right)$$
$$= (-1)^m \frac{d^m}{dx^m} L_{m+n}(x)$$

VI) Associated Lagendre Polynomials, $\phi_n^m(x)$, $n \geq m = 0, 1, 2, \ldots$

Range:

$$x \in [-1, 1]$$

Functions of Sturm-Liouville equation:

$$w(x) = (1-x^2)^m$$
$$f(x) = (1-x^2)^{m+1}$$
$$g(x) = 0$$

Differential equation:

$$(1-x^2)\phi_n^m{}''(x) - 2x(m+1)\phi_n^m{}'(x) + \lambda_n \phi_n^m(x) = 0$$

Eigenvalue:

$$\lambda_n = n(n+1) - m(m+1)$$

Eigenfunction:

$$\phi_n^m(x) = \frac{1}{2^n n!} \frac{d^{n+m}}{dx^{n+m}} (x^2-1)^n$$
$$= \frac{d^m}{dx^m} P_n(x)$$

VII) Associated Lagendre Functions, $P_n^m(x)$, $n \geq m = 0, 1, 2, \ldots$

Definition:

$$P_n^m(x) = (1-x^2)^{\frac{m}{2}} \phi_n^m(x)$$