UNIVERSITI TEKNOLOGI PETRONAS

FINAL EXAMINATION
JANUARY 2023 SEMESTER

COURSE	**:**	**FDM1023/FEM1023/FFM1023/FEM1083 - ORDINARY DIFFERENTIAL EQUATIONS/ENGINEERING MATHEMATICS II/MATHEMATICS FOR SCIENTISTS**
DATE	**:**	**4 APRIL 2023 (TUESDAY)**
TIME	**:**	**9:00 AM - 12:00 NOON (3 HOURS)**

INSTRUCTIONS TO CANDIDATES

1. Answer **ALL** questions in the Answer Booklet.
2. Begin **EACH** answer on a new page in the Answer Booklet.
3. Indicate clearly answers that are cancelled, if any.
4. Where applicable, show clearly steps taken in arriving at the solutions and indicate **ALL** assumptions, if any.
5. **DO NOT** open this Question Booklet until instructed.

Note :

i. There are **EIGHT (8)** pages in this Question Booklet including the cover page and the Appendix.

ii. **DOUBLE-SIDED** Question Booklet.

1. a. Solve the following initial value problem

$$\frac{dy}{dx} = \sec y \left(\frac{2x}{x^2+2}\right), \qquad y(1) = 0.$$

[4 marks]

b. Given the following first-order differential equation

$$y\,dx + (2xy - e^{-2y})dy = 0.$$

i. Verify the exactness of the equation.

[2 marks]

ii. Based on your answer in **Part 1(b)(i)**, solve the differential equation.

[7 marks]

c. Solve the following Bernoulli differential equation

$$x^2\frac{dy}{dx} + 3xy = \frac{6x^3y^{\frac{1}{3}}}{e^x}.$$

[7 marks]

2. a. Solve the following second-order differential equation

$$\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 16 = \sin(2x) + xe^{4x}.$$

Use the method of undetermined coefficients to find the particular solution.

[10 marks]

b. Solve the following non-homogeneous differential equation

$$x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} - 3y = x^3 \sin x.$$

Use the method of variation of parameters to find the particular solution.

[10 marks]

3. a. Evaluate

i. $$L\left\{(2t+8)U(t-7)+\frac{3\sinh 4t}{e^{2\pi t}}\right\},$$

[5 marks]

ii. $$L^{-1}\left\{\frac{s+7}{s^2+4s+9}\right\}.$$

[5 marks]

b. Use the Laplace transform to solve the following differential equation

$$\frac{d^2y}{dt^2}-5y=\cos t\,U(t-2\pi), \qquad y(0)=4,\ \ y'(0)=0.$$

[10 marks]

4. a. Given the following first-order differential equation

$$x\frac{dy}{dx} - y = x^3 + 3x^2 - 2x, \qquad y(1) = 2.$$

Solve the differential equation over the interval $[1, 5]$ using Heun's method with step size $h = 2$. Round off your results to five decimal places.

[8 marks]

b. Use the 4th order Runge-Kutta method with step size $h = 1.5$ to solve the following differential equation

$$\frac{dy}{dx} - \frac{6\sqrt{xy^2 + 8x}}{y} = 0, \qquad y(1) = 4.$$

over the interval $[1, 4]$. Round off your results to five decimal places.

[12 marks]

5. The heat distribution of a long rod with a length of 20 cm can be determined by solving the following heat-conduction equation

$$k\frac{\partial^2 T}{\partial x^2} = \frac{\partial T}{\partial t}$$

where $k = 0.835$ cm^{2}s^{-1} is the diffusivity constant. The initial and boundary conditions are

$$T(0, x) = 0,$$
$$T(t, 0) = 150^{\circ}C,$$
$$T(t, 20) = 80^{\circ}C.$$

a. Use the Crank-Nicholson method to generate the tridiagonal system of equations for the heat distribution at $t = 5$ s with step sizes $\Delta x = 4$ cm and $\Delta t = 5$ s.

[12 marks]

b. Based on your answer in **Part 5(a)**, solve the generated system using the Gauss-Seidel iterative method with initial guess, $T^{(0)} = 0$. Compute the numerical solutions for the first 2 iterations.

[8 marks]

- END OF PAPER -

APPENDIX

1. Trigonometric Identities

$$\cos(a+b) = \cos a \cos b - \sin a \sin b$$

$$\cos(a-b) = \cos a \cos b + \sin a \sin b$$

$$\sin(a+b) = \sin a \cos b + \sin b \cos a$$

$$\sin(a-b) = \sin a \cos b - \sin b \cos a$$

2. Variation of Parameters

Consider the differential equation

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = f(x).$$

The particular solution is

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x).$$

3. Laplace Transform of Basic Functions

$$L\{1\} = \frac{1}{s}$$

$$L\{t^n\} = \frac{n!}{s^{n+1}} \quad , \quad n = 1,2,3,\cdots$$

$$L\{e^{at}\} = \frac{1}{s-a}$$

$$L\{\sin kt\} = \frac{k}{s^2+k^2}$$

$$L\{\cos kt\} = \frac{s}{s^2+k^2}$$

$$L\{\sinh kt\} = \frac{k}{s^2-k^2}$$

$$L\{\cosh kt\} = \frac{s}{s^2-k^2}$$

4. Laplace Transform of Derivatives

$$L\{y'(t)\} = sY(s) - y(0)$$

$$L\{y''(t)\} = s^2Y(s) - sy(0) - y'(0)$$

5. First Translation Theorem

If $L\{f(t)\} = F(s)$ and a is any real number, then $L\{e^{at}f(t)\} = F(s-a)$.

6. Second Translation Theorem

If $L\{f(t)\} = F(s)$ and $a > 0$, then $L\{f(t-a)U(t-a)\} = e^{-as}F(s)$.

7. Fourth-Order Runge-Kutta Method

$$y(x_{i+1}) = y(x_i) + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

where

$$k_1 = f(x_i, y_i)$$

$$k_2 = f\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_1 h\right)$$

$$k_3 = f\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_2 h\right)$$

$$k_4 = f(x_i + h, y_i + k_3 h)$$

8. Crank-Nicholson Formula for Heat-Conduction Equation, $\frac{\partial T}{\partial t} = k\frac{\partial^2 T}{\partial x^2}$

$$-\lambda T_{i-1}^{l+1} + 2(1+\lambda)T_i^{l+1} - \lambda T_{i+1}^{l+1} = \lambda T_{i-1}^{l} + 2(1-\lambda)T_i^{l} + \lambda T_{i+1}^{l}$$

where $\lambda = \frac{k(\Delta t)}{(\Delta x)^2}$.