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APPROVAL PAGE

Classes and Properties of Exact Solutions to the Three-dimensional Incompressible Navier-Stokes Equations

By

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TITLE PAGE

Classes and Properties of Exact Solutions to the Three-dimensional Incompressible Navier-Stokes Equations

By

Gunawan Nugroho

A Thesis

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PERAK

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DECLARATION

Title of thesis	**Classes and Properties of Exact Solutions to the Three-dimensional Incompressible Navier-Stokes Equations**

I GUNAWAN NUGROHO hereby declare that the thesis is based on my original work except for quotations and citations which have been duly acknowledged. I also declare that it has not been previously or concurrently submitted for any other degree at UTP or other institutions.

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Dedications

O Thou, The Most Beautiful
Bestow me Thy Art to hidden paths
For men whilst know
Not against Thine

Not like the painting of a beautiful woman
Not like seductive love brought by a poet
The Navier-Stokes equations reveal our very deep of existence
Far beyond skin deep acts

This work is dedicated to my beloved mother and father Sutami and Tumaidi Guna Atmadja for their unending support in my passionate search of hidden structures behind the Navier-Stokes equations.

I also dedicate this work to my beloved wife and children for their patience and endurance.

And to any of those who are truly unexhausted to unravel nonlinearities.

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Abstract

The Navier-Stokes equations together with the continuity equation are one of the long standing problems in mathematical physics. They form a system of nonlinear partial differential equations that describe the fluid flow phenomena, whether laminar or turbulent. The nonlinearity of the equations is obscure which defies all conventional methods of analytical solution to the differential equations. The analytical methods are found to be very important to model physical phenomena. They form basic understanding of the phenomena at different circumstances, at least qualitatively. In addition to their physics, the analytical methods are also useful to find and extend the class of existence, uniqueness and regularity in the pure mathematics sense.

This thesis introduces new analytical methods of finding solutions of the incompressible Navier-Stokes equations. The work is based on the criteria of well-posed problem which is then solved by the proposed special classes of the solution either qualitatively or quantitatively.

Firstly, general qualitative properties of solutions to the three-dimensional incompressible flows are presented. The method is performed from the implementation of vector analysis into the energy equation with the consideration of zero rate energy. Trivial solution is obtained from any initial-boundary value problems. For the cases of non trivial solution, the analyticity of the solutions is assumed to investigate the triviality at intersection regions. Some physical consequences due to violation of the trivial solutions are also performed with the application of the vorticity equations, which are related to the onset of turbulence. Therefore, non trivial solutions will also represent turbulence whether they have singularity or not.

This hypothesis is supplemented by investigation on the solution in the special classes of $\vec{V} = \vec{\nabla} \times \Phi$ and $\vec{V} = \vec{\nabla}\Phi + \vec{\nabla} \times \Phi$ of the three-dimensional incompressible Navier-Stokes equations. Analysis is taken using the vorticity equations rather than the original Navier Stokes equations based on qualitative mathematical work. Results

show that the corresponding problem admits a unique and regular solution because the original problems can be transformed to class of linear parabolic and elliptic equations.

The first analytical solution is then produced using the four components coordinate transformation $\xi = kx + ly + mz - ct$. While, the second solution is produced using the three components coordinate transformation $\xi = ly + mz - ct$. Velocity vector in the solutions is based on the relation $\vec{V} = \vec{\nabla}\Phi + \vec{\nabla} \times \Phi$ where Φ is a potential function that is defined as $\Phi = P(x,\xi)R(\xi)$. The potential function is firstly substituted into the continuity equation. The solution for R is produced using a certain mathematical condition and the resultant expression is used sequentially in the Navier-Stokes equations to reduce the problem to the class of nonlinear ordinary differential equations in P terms. Here, more general solutions are also obtained based on the particular solutions of P. The two solutions are based on a zero and constant pressure gradients which are given to illustrate the applicability of the method.

The third analytical solution utilises a potential function in the form $\Phi = P(x,y,\xi)R(y)S(\xi)$ with the application of the transformed coordinate $\xi = kz - \varsigma(t)$. In this solution, the pressure term is presented in a general functional form. The solutions for R and S are obtained by imposing a certain mathematical condition. General solutions are then obtained based on the particular solutions of P where the equation is reduced to the form of linear differential equation. A method for finding closed-form solutions for general linear differential equations is proposed and uniqueness of the solution is proved and regularised.

The fourth analytical solution is derived using the vorticity equation. The solution is produced by implementing a potential function in the form $\Phi = P(x,y,\xi)R(y)S(\xi)$ with the application of the transformed coordinate $\xi = kz - \varsigma(t)$. The pressure is then solved by applying the velocity vector into the Navier-Stokes equations to complete the solutions. Two examples are given to illustrate the applicability of the theorem. The uniqueness of the solution is also proved.

Validation against two laminar flow experiments and three different turbulent flow cases including numerical case are carried out and reported in this work. The flow cases used in the validation are laminar jet flow, turbulent jet flow, boundary layer flow, turbulent channel flow and combustion. Generally, the solution is able to follow the trends in the corresponding cases. Although the analytical solution is derived for non-reacting flows, it proved capable of reproducing trends of cases including combustion.

Abstrak

Persamaan-persamaan Navier-Stokes bersama persamaan keselanjaraan adalah salah satu persoalaan dalam bidang fizik matematik. Persamaan-persamaan ini merupakan persamaan pembezaan separa non-linear yang digunakan untuk menjelaskan fenomena berkenaan dengan mekanik bendalir sama ada dalam keadaan lamina atau bergelora. Oleh kerana persamaan-persamaan tersebut mempunyai ciri-ciri non-linear yang tinggi, maka kaedah-kaedah analitis yang biasa digunakan untuk mendapatkan penyelesaian tidak berkesan. Walaubagaimanapun kaedah-kaedah analitis ini adalah penting untuk mendapat gambaran dan persefahaman yang jelas terhadap fenomena fizikal mekanik bendalir. Kaedah-kaedah analitis ini mejadi asas kepada persefahaman dalam pelbagai keadaan secara kualitatif. Selain itu, kaedah-kaedah analitis ini juga amat berguna untuk mendapatkan dan melanjutkan kelas perwujudan, keunikan dan keteraturan ilmu matematik yang terhasil dari persamaan-persamaan Navier-Stokes.

Thesis in memperkenalkan kaedah analitis yang baru untuk mencari penyelesaian kepada persamaan-persamaan Navier-Stokes dalam keadaan aliran bendalir yang tidak bermampat. Penyelidikan ini menggunakan hypothesis bahawa terdapat kewujudan penyelesaian kepada persamaan-persamaan Navier-Stokes. Seterusnya penyelesaian ini dicari dengan menggunakan kelas-kelas matematik yang khas yang diajukan secara kualitatif mahupun kuantitatif. Kelas-kelas inilah yang diperkenalkan dalam penyelidikan ini.

Pertama, ciri-ciri penyelesaian yang kualitatif kepada persamaan-persamaan Navier-Stokes dalam tiga dimensi dengan keadaan aliran tak bermampat dibentangkan. Kaedah ini merangkumi penggunaan analisa vektor dalam persamaan tenaga di mana kadar tenaga dianggap sifar. Penyelesaian ringkas boleh dihasilkan dari mana-mana masalah matematik yang mempunyai nilai awal batasnya. Bagi penyelesaian yang kompleks, penyelesaian yang ringkas boleh didapati daripada bahagian-bahagian persamaan yang tertentu. Walaubagaimanapun terdapat beberapa

masalah fizikal yang timbul dalam usaha mencari penyelesaian ringkas. Masalah-masalah ini boleh diatasi dengan menggunakan persamaan-persamaan pusaran yang berkaitan dengan permulaan fenomena pergeloraan. Oleh kerana itu, penyelesaian kompleks juga memberi penjelasan kepada pergeloraan bendalir sama ada kewujudan singulariti atau tidak.

Hipothesis ini dikukuhkan dengan menyelidik penyelesaian pada kelas-kelas khas iaitu, $\vec{V} = \vec{\nabla} \times \Phi$ dan $\vec{V} = \vec{\nabla}\Phi + \vec{\nabla} \times \Phi$ dalam persamaan-persamaan Navier-Stokes dalam tiga-dimensi bagi keadaan aliran bendalir yang tidak bermampat. Analisa diambil dari persamaan-persamaan pusaran berbanding dengan persamaan-persamaan Navier-Stokes yang asal yang berdasarkan analisa matematik secara kualitatif. Keputusan yang didapati menunjukkan bahawa penyelesaiannya adalah unik dan teratur. Ini adalah kerana, persamaan-persamaan asal Navier-Stokes boleh ditransformasikan kepada kelas persamaan linear parabola dan ellispsis.

Dengan itu, maka penyelesaian analitis yang pertama boleh didapati dengan menggunakan transformasi empat koordinat komponen vector $\xi = kx + ly + mz - ct$. Manakala, penyelesaian yang kedua boleh didapati menggunakan transformasi tiga koordinat komponen vector $\xi = ly + mz - ct$. Vektor halaju yang diperolehi daripada penyelesaian berasaskan persamaan $\vec{V} = \vec{\nabla}\Phi + \vec{\nabla} \times \Phi$ di mana Φ adalah fungsi potensi yang boleh dijelaskan dalam bentuk $\Phi = P(x,\xi)R(\xi)$. Fungsi potensi ini digantikan dalam persamaan keselanjaraan terlebih dahulu. Penyelesaian untuk R dihasilkan dengan menggunakan keadaan matematik yang tertentu dan persamaan yang didapati dari keputusannya digunakan secara berturutan dalam persamaan-persamaan Navier-Stokes untuk meringkaskan masalahnya kepada kelas persamaan pembezaan nonlinear dalam bentuk P. Di sini, penyelesaian yang lebih am boleh didapati berasaskan penyelesaian tertentu P. Kedua-dua penyelesaian ini adalah berasaskan kecerunan tekanan bersamaan dengan sifar atau dengan kecerunan tekanan yang mempunyai nilai tetap. Ini menunjukkan bahawa kaedah ini amat berkesan.

Penyelesaian analitis yang ketiga pula menggunakan fungsi potensi dalam bentuk $\Phi = P(x,y,\xi)R(y)S(\xi)$ dengan penggunaan transformasi koordinat $\xi = kz - \varsigma(t)$. Menggunakan penyelesaian ini, tekanan dipaparkan dalam bentuk fungsi yang am. Penyelesaian untuk R dan S dihasilkan dengan menggunakan keadaan matematik yang tertentu. Dengan itu, maka penyelesaian am didapatkan berasaskan penyelesaian tertentu P di mana persamaannya diringkaskan kepada bentuk persamaan perbezaaan linear. Maka ini adalah satu kaedah untuk mencari penyelesaian dalam bentuk tertutup untuk persamaan pembezaan am. Selain itu, dengan kaedah ini keunikan dan keteraturan penyelesaian juga boleh dibuktikan.

Penyelesaian analitis yang keempat dilanjutkan dari persamaaan pusaran. Penyelesaian ini dihasilkan dengan menggunakan fungsi potensi dalam bentuk $\Phi = P(x,y,\xi)R(y)S(\xi)$ dengan koordinat yang telah dijalankan transformasi $\xi = kz - \varsigma(t)$. Tekanan pula diselesaikan dengan menggunakan vector halaju dalam persamaan Navier-Stokes untuk menyempurnakan penyelesaiannya. Dua contoh telah digunakan untuk membuktikan kebolehgunaan theorem ini. Keunikan penyelesaian ini juga telah dibuktikan.

Sebagai pengesahan kepada penyelesaian yang dihasilkan melalui teknik ini, keputusan dari dua eksperimen aliran lamina, tiga eksperimen aliran bergolak dan penyelesaian numerikal telah dibandingkan dengan penyelesaian analitis yang dihasilkan. Kes aliran yang digunakan untuk tujuan pengesahan adalah aliran jet lamina, aliran jet dengan pergolakan, aliran lapisan batas dan kajian pembakaran. Secara amnya, penyelesaian analitis yang dihasilkan menepati keputusan kes-kes tersebut. Walaupun penyelesaian ini dihasilkan untuk aliran yang tidak bertindakbalas, penyelesaian ini telah membuktikan bahawa ia juga boleh digunakan untuk kes-kes seperti kajian pembakaran.

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Nomenclatures

General Symbols

$\vec{V}$	Velocity vector
u, v, w	Velocity components of x, y, z directions
$\vec{\omega}$	Vorticity
p	Pressure
E	Energy
ρ	Fluid density
υ	Kinematic viscosity
$\vec{\nabla}$	Vectorial derivative
$\mathbb{R}^n$	n-dimensional spatial coordinate
Ω, x, y, z	Spatial coordinates
T, t	Time coordinates
ξ, η, ζ, τ	Functional transformed coordinates
$\vec{n}$	Unit vector
S	Surface domain

Symbols in Chapter 3

Ξ_i	Simple region
Ξ_{ai}	Non simple region
A	Area
F	Total sum of simple and non simple domains
m	Number of simple sub domains
n	Number of non simple sub domains
$z(m)$	First functional statement of F

$g(n)$	Functional statement of non simple sub domains
$h(m)$	Functional statement of simple sub domains
j	Second functional statement of F, where $z < j$
$f(x)$	Solution at the intersection regions
B	Limit convergence in the first intersection region
C	Limit convergence in the second intersection region
Ψ	A particular sub region
Φ	Potential function
Λ	Functional coordinates of Φ
Γ	Functional parameters of Φ
$g(x,y,z,t)$	Solution from Λ
L^p	Lebesque space
C^n	n-differentiable functions
Supp	Supprenum (maximum value)
L	Linear operator
$\beta, \varepsilon, \alpha$	Arbitrary constants
ϑ	Weak and strong solution
s	Time and spatial values
M	Constant from Poincare inequality
$W^{k,p}$	Sobolev space
H^k	Hilbert space

Symbols in Chapter 4

ξ	Explicit transformed coordinate
k, l, m, c	Constants of coordinate transformation
A_i, B_i, C_i, D_i	Constants due to coordinate transformation
P, R, S	Components of potential function decomposition
φ, α	A particular solution of R
$a_i, b_i, c_i, d_i, e_i, f_i, g_i, h_i, i_i$	Constants with respects to x
l_i	Functional due to P ratio

Q	Functional transformation of P_x
G	Functional transformation of P
N	Functional transformation of Q_x
U, W	Particular components of P
η, A	A particular solution of W
$r_i(x)$	A functional statement of U
γ_i	Pressure gradients
χ	A particular solution of P
λ	Multiplier
D	A function appears due to decomposition of A
κ	An arbitrary function
n_i	Functional solution of W
q, k_i, F, H, s_i	Decomposition function of W
x^*, y^*, z^*, t^*	Specific points in the solution

List of Publications

1. Nugroho G., Ali A.M.S and Abdul Karim Z.A., On a Special Class of Analytical Solutions to the Three-Dimensional Incompressible Navier-Stokes Equations, *Applied Mathematics Letters 22*, 2009, Elsevier, ISSN: 0893 – 9659, pp. 1639 – 1644.

2. Nugroho G., Ali A.M.S and Abdul Karim Z.A., A Class of Exact Solution to Three-Dimensional Incompressible Navier-Stokes Equations, *Applied Mathematics Letters*, 2010, 10.1016/j.aml.2010.07.005, Elsevier, ISSN: 0893 – 9659.

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5. Nugroho G., Ali A.M.S and Abdul Karim Z.A., Triviality in Boundary Value Problems of Three-Dimensional Incompressible Navier-Stokes Equations and its Violation Based on the Condition of Zero Rate Energy, *Far East Journal of Applied Mathematics*, 2010, ISSN: 0972 – 0960, accepted.

6. Nugroho G., Ali A.M.S and Abdul Karim Z.A., Exact Solution to the Three-Dimensional Incompressible Navier-Stokes Equations through the Vorticity Equations, *IMA Journal of Applied Mathematics*, 2010, Oxford, ISSN: 0272-4960, submitted.

7. Nugroho G., Ali A.M.S and Abdul Karim Z.A., A Short Review on the Analytical Methods in Fluid Dynamics Research: The Importance of Exact Solutions, *Proceeding of ICPER*, 2010.

Chapter 1
Introduction

1.1 Background

The importance of the Navier Stokes equations comes from their wide application for different kind of flows, ranging from thin films to large scale atmospheric even cosmic flows. However, Navier-Stokes equations are nonlinear in nature and it is difficult to solve these equations analytically. In order to perform this task, some simplifications are elucidated, such as linearisation or assumptions of weak nonlinearity, small fluctuations, discretisation, etc.

Despite the concentrated research on the Navier Stokes equations, their universal solution is not achieved. The full solution of the three-dimensional Navier-Stokes equations remains one of the open problems in mathematical physics. Computational Fluid Dynamics (CFD) approaches discretise the equations and solve them numerically. Although such numerical methods are successful, they are still expensive and there must be approximation errors associated with them.

The development of high speed computers eventually makes discretisation methods more advance than the others and it enables the numerical treatment of turbulent flow. Solution of turbulent flows mainly depends on solving the Navier Stokes equations and using ad-hoc models to close the solution. The numerical approaches are Reynolds Averaged Navier Stokes (RANS) which provides averaged solution of the flow, Large Eddy Simulation (LES) which solves the big scales and model the small ones and Direct Numerical Simulations (DNS) which solve all the flow scales. With respect to the computational cost, DNS is the most expensive model and it is still limited to small scale research problems. LES guarantees more economical computational time as compared to DNS and the results are not much different than DNS results when appropriate subgrid scale (SGS) models are used [1]. The cost of computation depends also on the dimension of the case and on the coupling with other equation as well, like in turbulent reacting flow. Hence, a better understanding of the corresponding phenomena is still needed since those models do not provide accurate prediction for complex flows [2].

There are numerous researches concentrating on formulating efficient numerical schemes in solving Navier-Stokes equations, such as the recent work as described by [3] and [4]. However, the computational costs are still expensive for handling accurate numerical simulations except for simple problems in engineering limited to small scale problems, concerning that full solution must describe the evolution of the physics in pointwise. Besides, numerical solutions have well-known weakness in boundary layer regimes near solid boundaries and interface of turbulent-nonturbulent regimes in which weak solutions are not unique [5]. It is known that in finite time interval, the solution of the Navier-Stokes equations may either blown up or split up, losing its regularity, and beginning to form branches [6,7]. In particular, numerical iterations also make the blow up investigation difficult due to the rapid numerical fluctuations [8]. In fact, depending on the values of the relevant parameters, a stationary boundary value problem can have a unique solution, several solutions, or even no solutions at all.

One of the important problems in the theory of partial differential equations (PDEs) is finding and studying classes of integrable equations which have explicit solutions, specially, closed form solutions. After the Blasius famous work on the exact solution for the two-dimensional (2D) boundary layer equations proposed by Prandtl, similarity solutions of linear and nonlinear boundary-value problems became more common in the literature. The work for finding solutions of ordinary differential equations (ODEs) and partial differential equations by symmetry reductions date back to the famous Sophus Lie works.

However, the progress achieved in existence-uniqueness-regularity theory for the Navier-Stokes equations somehow causes explicit solutions slowly losing their important role. The efforts are facing some difficulties, especially in higher dimension. The existence and uniqueness classes for the Navier-Stokes equations are harder to see and difficult to prove [9]. For instance, there are several fundamental open problems on them.

For many cases which the heroic attempt on rigorous mathematical works remains elusive, the exact solutions give the promising way to detect significant features of nonstationary and singular evolution, such as gas dynamics spherical waves and shockwave phenomena. Therefore, it was not surprising that the Navier-Stokes equations are among the first applications of fresh ideas and new developed methods of the exact explicit integration [10].

However, it is important mentioning what is meant here by exact solutions. One well known definition of exact solution is to detect the explicit solutions expressed in terms of elementary or, at least, known functions of mathematical physics. Exact solutions may also be defined as functions generated from some ordinary differential equations or from reduced partial differential equations which order is lower than the original problems.

Since the difficulty to find rigorous proof of existence, regularity and uniqueness of a certain class of differential equations is overcome by exact solutions, most likely they will continue to play a decisive role. They always provide us with fundamental patterns in order to generate more physically reasonable solutions, such as specific asymptotics. Also, exact solutions of nonlinear models are significant in the theory of nonlinear evolution equations. Exact solutions are often demanded in the development general existence-uniqueness and asymptotic theory. The role of exact solution in revealing an optimal description of local and global existence functional classes, uniqueness classes and generic asymptotic behavior is an inevitable fact as shown in many examples of nonlinear models [11].

Their specific space-time structure sometimes gives hindsight of some crucial features in order to develop the new methods and tools, which are required for constructing general solutions. In the theory of parabolic reaction-diffusion equations, there exist well known cases where the method of nonlinear transformation determines the correct rescaled variables. In this case, exact solutions give guidance in terms of which the maximum principle can be applied to extend regularity properties to more general ones [12].

Therefore, the problem of searching the classes of exact solutions of the full Navier-Stokes equations is highly demanding from a practical viewpoint, as has been described in the literature [13]. Exact solutions also facilitate a theoretical understanding, paving the way to global solutions. They may help explain the issue of global smoothness in time [14]. The solutions may be examined as models for turbulence [15]. The specific example of this is a particular vortex solution, which is significant in the development of turbulence theories [16].

Unfortunately, only a few analytical works are present in the literatures. Some of them are investigated by different researchers independently as collected over the range of references [17,18,19]. As in most cases, exact solutions are examined only in the special conditions which the nonlinearity are weakened or even removed from the analysis. The type of the simplified analysis is steady and unsteady Couette and Poiseuille flows which nonlinear terms remove permanently [20]. The other less known example is Beltrami flows where the nonlinear terms are nonzero in the Navier-Stokes equations but fade in the vorticity equations [21].

More sophisticated analysis of the Navier-Stokes equations is also conducted and gives more insight to the problems. One of them is the transformation of the Navier-Stokes equations to the Schrodinger equation, performed by application of the Riccati equation [22]. It has good prospects since the Schrodinger equation is linear and has well defined solutions. The method of Lie group theory is also applied in order to transform the original partial differential equations into ordinary differential systems [23]. It is concluded that an approximate series solution is obtained. The same route is taken by Meleshko [24] and by Thailert [25], in transforming the Navier-Stokes equations to solvable linear systems. On the other hand, less popular methods, such as the Hodograph-Legendre transformation, have also been applied to reduce the original problem to one more tractable, and thus closer to the goal of obtaining analytical solutions [26]. The method of introducing special solutions for velocity has also been investigated [27,28].

The lack of development in analytical analysis to solve fluid dynamics problems is not without reason. In the past, analytical analysis was used in order to obtain closed-form solutions, which many of them form our basic intuition of fluid phenomena. This aspect is now considered as somewhat obsolete by the enormous increase in capacities of numerical computing. Up to this point, many of fluid flow problems are not yet solved by closed form solutions, also most of interesting and important problems are either unsolvable or only tractable to numerical simulation after some appropriate evaluation from analytical solutions. It is known that for some time the development of the capabilities of numerical computations will depend on, or at least, connected to the development of analytical methods. Analytical methods are expected to generate simpler, adequate and significant models important to numerical simulation and very efficient for the stiff and numerically difficult problems [5].

One example for the controlling perturbation quantities is such as Mach number for small parameter, or for a large parameter such as the Reynolds number, which is important for the asymptotic modeling. If the Navier-Stokes equations describing a precise flow problem can be expressed in elementary function from analytical methods, then the full solutions may be generalised. At least if such that one of the parameters or variables is known to be small or large, then, general solutions can be approached by perturbation quantities. Solutions will approach a limit as the perturbation quantity approaches zero or infinity and thus resulted in asymptotic functions. The result can often be improved by expanding in a series of successive approximations from the first term of which is the limiting solution as an asymptotic series or expansion [29].

1.2 Problem Statements

The global in time continuation to the three-dimensional incompressible Navier-Stokes equations remains the major unsolved problem in the mathematical fluid mechanics. At the physical viewpoint, the debate is as if such singularity exist, it will be associated to turbulence by provoking that we have anything regular in two-dimensional flow cases, which turbulence is three-dimensional phenomena. This hypothesis, however, have problems since it is witnessed that all turbulence is bounded in nature. This issue somehow drives qualitative mathematical analysis of the viscous incompressible flows toward the enormous development, but only a little interest pointed to the analytical methods which provide a quantitative understanding.

Therefore, finding the appropriate methods for analytical treatment of the full set of incompressible Navier-Stokes equations is very important task from theoretical and practical point of view. Specifically, at the theoretical side, the results then can be extended to set and justify the posedness problems (existence-uniqueness-regularity) of fluid dynamics.

1.3 Research Objectives

The main objective of this research is to find classes of exact solutions to the three-dimensional incompressible Navier-Stokes Equations. Related matters as the physical and mathematical theory of the solutions are also developed. Second, the physical theory of the incompressible Navier-Stokes equations is investigated to provide the transition to turbulence as well as mathematical theory of the rigorous solutions. Since, the exact solutions are expected to provide fast calculations and models important to numerical methods, the generating solutions are then validated with the existing experimental and n umerical data from the literatures. The validated cases include laminar and turbulent jet flows, turbulent boundary layer flows, turbulent channel flow and also trivial cases including combustion.

1.4 Scope of Works

The scope of the current investigation is to solve the three-dimensional incompressible Navier-Stokes equations and continuity equation by self developed analytical techniques. This current research is conducted by implementing the classical Cartesian coordinate system. However, the novel techniques developed in this research are also applicable to any coordinate systems. The phenomenological issue related to the turbulence is also addressed following four preliminary validating processes which are discussed in a concise manner for each case.

1.5 Research Methodology and Work Outline

The work is divided in two parts. First, the contribution is to the abstract analysis on the physical and mathematical theory to the incompressible Navier-Stokes equations. As for the physical theory, the elementary vectorial analysis is implemented to a simple energy equation. The divergence theorem will generate trivial results for general situations which the violation of the triviality is lead to a possible onset of turbulence due to energy accumulation and dissipation. For the mathematical theory, the analysis is stressed on the rigorous proof on the existence and uniqueness of the proposed classes of the solutions. The original problem is mapped into different parameter space which is linear, the remapping process also leads to the linear differential equations which have well defined proof for existence, regularity and uniqueness of the solutions.

Second, a special class of solutions to the three-dimensional incompressible Navier Stokes equations is investigated further. A potential function and a transformed coordinate are proposed, and the three equations are altered into simpler equations in terms of the potential function and the transformed coordinates. The proposed class of solutions is first substituted to the continuity equation and the resultant expression is employed sequentially into the Navier-Stokes system to find full solutions. Then, particular analytical solutions are obtained and extended to the more general form.

This work is continued to a nontrivial coordinate relation with respect to time. It will serve to the more general function, which analysis in time coordinate can be chosen as any function whether describing blown up solutions or not. The extended analysis to general functional relation due to pressure gradient is also performed. The procedure is also developed in a sense that only one spatial and time coordinate transformed to a single coordinate which is supplemented by a new proposed method to find closed-form solutions for general second order ordinary differential equations.

It is interesting to mention that solutions of the vorticity equations drive towards the collection of exact solutions to the Navier-Stokes equations. The vorticity representation is reasonable and physically clear, at least for incompressible flows [30,31]. The advantage to consider vorticity equations is its capability to remove pressure relation due to vector identity. The pressure hessian then can be calculated from the divergence operation after velocity relation is obtained. The method for finding particular solutions is also more developed in this section if the derivation is brought into the first order differential equations. Finally, preliminary validation work is conducted to test the applicability of the solutions.

1.6 Thesis Outline

This thesis consists of six chapters with references. This chapter (chapter 1) presents outline about the importance contribution of analytical research, especially exact solutions to fluid dynamics. The problem statements concerning to the specific area are explained briefly. The research objectives then proposed followed by methodology and scope of works including work outline.

Chapter 2 deals with the literature reviews of the previous researches related to this work. Firstly, some aspects of exact solutions in the fluid dynamics research are reviewed extensively. The subject then followed by the review of turbulent flows which is arranged such that it is clear as to what, why and how analytical research will significantly contributes to resolve turbulent flow problems.

Chapter 3 is deals with the contribution to the theory of solutions to the three-dimensional incompressible Navier-Stokes equations. The contributions are in the abstract analysis which the results represent physical and mathematical properties with respects to the proposed classes of the solution.

Chapter 4 deals with the contributions to the methods of generating exact solutions to the three-dimensional incompressible Navier-Stokes equations with respect to the proposed class of the solution. Analysis is divided into four sections based on the generality of the problems. Each section reveals its own specific problems and difficulties concerning to the coordinate transformations, pressure and time relations.

Chapter 5 deals with the validation cases of the solution. The validation cases are laminar and turbulent flows including combustion. Each case is discussed concisely from physical point of view.

Chapter 6 deals with the summary of works and discussion about the possible future works which relevant to the general problems of fluid dynamics research.

Chapter 2
Literature Reviews

2.1 Some Aspects of Exact Solutions

The closed form solutions of the integrable systems, even with infinite number of degrees of freedom, always show regular and organised behavior. Solitons in the systems described the Korteweg de Vries and Schrödinger equations [32,33] are the outstanding examples. This kind of solution is interpreted as a mutual interaction among their nonlinearity such that the solution becomes perfectly regular. Also, the Burgers equation which integrable and shows random behaviour only under influence of random force [34]. These examples reveal the behavior of nonlinear systems to random forcing and should be distinguished from problems involving self excitation turbulence. The Navier-Stokes equations at a certain large Reynolds number have the property to initiate a randomisation, which are not fully understood. There is no universal agreement of what is inside turbulent flows. It can be stated almost everything including the direct experimental results of numerical and laboratory, which can be obtained from the first principles, the Navier-Stokes equations.

The theory of the Navier-Stokes equations constitutes a central problem in recent development of mathematical physics. These equations are physically well accepted model for description of most fluid flow phenomena and much effort has been placed by mathematicians, physicists, engineers, meteorologists and others. However, many problems are still waiting to be mathematically and physically resolved at the front of science. On the mathematical side, the Navier-Stokes equations are model for the investigation of nonlinear phenomena and nonlinear equations which are not well-developed enough [35]. The Navier-Stokes equations and the related Euler equations issue problems in nonlinear analysis, well-posedness and nonlinear dynamics.

One of the main issues is an analytical treatment of the corresponding fluid equations which relates to the attempt of generating function appear in the solutions. Such analytical solutions are usually represented by elementary functions known to mathematical physics. Elementary functions arise from many resources, in most cases they appeared naturally as a result of the solutions of linear ordinary differential equations. This is true, for example, of first order equations, where exponential functions describe the essential decaying nature near the infinity. Such nature can also be described by the Bessel function and the hyperbolic function that occurs in more complex equations. In fluid dynamics, this situation always arises at a slight corner where the equations are elliptic, as for the biconvex airfoil [36]. When the problem is confidently described as the correct diagnosis, it may be possible to render the solution uniformly valid simply by replacing the exponential functions with near-integral powers. In inverse coordinate expansions for viscous flow, these seem often to be required to ensure exponential decay of velocity.

On the other hand, it may be found that the straightforward solutions represented by elementary functions are not uniformly valid throughout the flow field. The best known example is separated viscous flow at moderate Reynolds numbers, where viscous motion fails near the surface. It is widely interpreted that the kinetic energy is not enough to overcome friction and adverse pressure gradient. This is usually supplemented by the boundary layer approximation from the modified Navier-Stokes equations. Not only does the first approximation breaks down locally in such cases, but the difficulty is embedded in higher approximations such that in the region of non-uniformity the solution grows worse rather than better [5,37]. In other problems their source is even more obscure and these somehow belongs to the analysis of singular problems.

However, because the solution is affected by a change of coordinates, the question was implicit when it is observed that investigations of the semi-infinite flat plate are carried out in other coordinate systems rather than the traditional Cartesian coordinates. Suppose that the entire boundary layer analysis is repeated in a different coordinate system, that is, the Navier-Stokes equations are written in the new system, the stream function and coordinate normal to the surface are modified by a certain well defined factor. Physically, the resulting boundary layer equation should be solved subject to zero velocity at the surface and matching with the basic inviscid flow in outer layer. It will be found that although the boundary layer is invariant, the outer layer region is not, so that it represents an altered flow field. Different coordinate systems yield boundary layer solutions that are identical at the surface and differ only negligibly within the boundary layer. It shows that the skin friction is invariant but may differ significantly outside the considered boundary layer.

The search for special coordinates was inspired by the fact that if certain nonlinear partial differential equations are approximated by different equations or even set of equations, the solution then is the exact solution of the original equation mapped to different coordinates [26,38]. This, together with other considerations already mentioned, suggested that special coordinates may be preferable for some physical cases when the full Navier-Stokes equations are used, and this has been seen to be true.

For boundary layer solution, it has been customary to disregard the solution outside the boundary layer region, where it can be replaced by the matched function of inviscid flow [39]. However, it may not be necessary to repeat the boundary layer solution when the coordinates are changed. The advantage of the changes can be investigated by seeking a special coordinates system in which the boundary layer solution attaches to the outer flow. When any convenient solutions are calculated in any coordinate systems, its counterpart in any other system is given by a simple rule.

The transformations used in the methods suffer despite the evident utility from the objection that they must usually be applied arbitrarily and blindly, with no understanding of the mechanism involved. It also may not be sufficient when neither the nature nor the location of the singularity that limits solutions is known. It also goes without saying that one does not always achieve such significant improvement as in the preceding examples. With further insight into the source of blow up, the equations can be processed more rationally and more successfully. Thus preliminary knowledge of the location of the singularity may be considered, though not its character. It often happens in mechanics that a power series expansion having physical significance only for positive real values of the perturbation quantity is restricted by a singularity elsewhere in the complex plane, usually on the negative axis, and if the variables have been chosen in the most natural way. This situation exists may be known from fundamental considerations, or be suggested by a rational fraction as in [40], or merely be suspected.

Singularity problems also arise frequently in fluid mechanics, since they have been studied increasingly in recent literature as the requisite mathematical techniques were developed [29,41]. Understanding is gained into even some classical problems by recognising their singular nature. These efforts are largely devoted to the rigorous proof of the existence and uniqueness of the solutions. Mathematical justification of these procedures is fairly complete in two-dimensional cases, but still open for three-dimensional cases. Therefore no precise statements can be made as to when the solutions can be applied, as to which is preferable in a given problem, or as to how analytical methods are related to each other. Nevertheless, in some cases, ordinary differential equations can be adopted as simple models to demonstrate the important points [35]. The nonlinear term in Burger equation for example, lead to produce regions of steepened velocity gradients, which implies to a transfer of excitation from large scale to the small scale of the velocity field. The velocity field is reduced to a sparse collection of shocks, with smooth and simple variation between fronts if the Reynolds number is high. This feature make Burgers' equation a valuable tool.

However bad the solution is, various technological and complex fluid flow problems are resolved via massive computations and often ad hoc arguments to lead to their results. However, only a few works are conducted to justify the assumptions on mathematical arguments. It is very important that a rational, consistent, approach be developed to ensure its validity. Obviously, until this problem is settled, the value of the results of such a computation may be questioned. In fact, it is necessary to understand that actually the numerical methods and exact solutions both are useful and complementary.

Aside from any physical considerations, fluid flow phenomena are most inherently three dimensional and time dependent. Thus, an enormous amount of information is required to completely describe such flows. Fortunately, it usually require something less than a complete time history over all spatial coordinates for every flow property. Thus, for a given flow conditions, the following question must be issued. Given a set of initial and boundary conditions, how do the physically meaningful properties of the flow predicted. What properties of a given flow are meaningful is generally dictated by the application. However, for the simplest applications, it may require only the skin friction and heat-transfer coefficients [43].

Though we have a set of deterministic differential equations for describing fluid flow phenomena, believed consist of all of turbulence known to experiments, most of our understanding about turbulent flows were gained from the observations of controlled numerical and physical experiments. There is only a little attempt of theoretical analysis of the Navier-Stokes equations in turbulence. However, there are several routes to do this indirectly or by using of the Navier-Stokes equations and their consequences. In the sense that manipulating the Navier-Stokes equations and their consequences will enable us to recognise the dynamically important quantities and physical processes involved. In other words, the modified Navier-Stokes equations and their consequences describe what quantities and relations should be studied. So far this can be done mostly experimentally, but this kind of guiding should also be useful theoretically [44].

More complex phenomenological considerations might require detailed knowledge of energy spectra, turbulence fluctuation magnitudes and scales. Certainly, it should be expected that the complexity of the mathematics needed for a given application to increase as the amount of required flow field detail increases. On the other hand, if the only required property is skin friction for an attached flow, a simple mixing-length model may suffice [45]. Such models are well developed and can be implemented with very little specialised knowledge.

Generally, all flows of practical engineering interest are turbulent. Thus, once the question of how much detail needed is answered, the level of complexity of the model follows. In the spirit of Prandtl, Taylor and Von Karman, the engineers will mostly prefer to use simple approach to reach their practical results. Phenomenologically, turbulence is characterised by the presence of a large range of excited length and time scales. The irregular properties of turbulence are strongly opposed to laminar motion, because the flow moves in smooth laminae, or layers.

2.2 Challenges of Turbulent Flows

Any turbulent flow is maintained by an external source of energy produced by one or more mechanisms [46]. The mechanisms in maintaining and sustaining turbulence are observed to be strongly related to the way of laminar and transitional flows become turbulent. Apart from instabilities, turbulent flows can be produced by brute forces, or by applying external force of both in real experimental set up and in computations by adding some forcing in the right hand side of the Navier-Stokes equations. For example, one of the simplest kinds of turbulent flow, quasi homogeneous and isotropic, can be generated by moving a grid through a fluid medium or using a grid in a wind tunnel [47], oscillating them in a water tank, or forcing turbulent motion by electromagnetic forces, like in plasma, liquid metals or even electrolytes [48]. A similar aspect of turbulent flow can be numerically produced by adding random or deterministic forces which acting to the right hand side of the Navier-Stokes equations. An important point is that the nature of additional force is secondary in establishing and sustaining a turbulent flow, which means that the Reynolds number is sufficient to create turbulent flows. In addition, the additional force would likely be in the large scales.

Another important point is that the forcing does not have to be random. Even if a turbulent flow is produced by random forcing, the primary role of such forcing is to supply energy to the flow and to trigger the intrinsic mechanisms of self randomisation of turbulent flow [49]. This is apparently to be a reason to differentiate turbulent flows, the one which occur naturally and the other generated by external random or deterministic source. They share similar qualitative properties and also quantitatively similar in several aspects. It is already observed that the flow produced by deterministic forces in the right hand side of the Navier-Stokes equations is not random for small Reynolds number. For the flow produced by random forcing, though random, it is in many aspects not the actual turbulence meaning that it is rather trivial, and there is mostly no interaction between its modes, in the sense of Fourier analysis. It is not caused by the state of transition from laminar to turbulent flow. This problem is related to the old philosophical question on whether flows become or whether they are just turbulent. It can be from any initial state including a turbulent one, such as random initial conditions in direct numerical simulations of the Navier-Stokes equations [50].

Turbulent flows always occur when the Reynolds number is large. Careful analysis of solutions to the Navier-Stokes equations of its boundary layer forms shows that turbulence develops instability of laminar flow. To analyse the stability of laminar flows, virtually all methods begin by linearising the equations of motion [51]. Even though some results can be obtained in predicting the instabilities which lead to turbulence with linear theories, the inherent nonlinearity of the Navier-Stokes equation most likely precludes an analytical description of the actual transition process. For viscous fluid, the instabilities result from interaction between nonlinear inertial terms and viscous terms of the Navier-Stokes equations. The process is very complex because it is rotational, three dimensional and time dependent. The nature of turbulence strong rotational behavior is closely correlated with its three-dimensionality.

Turbulence also consists of a wide spectrum of scales ranging from largest to smallest. In order to visualise a turbulent flow with a spectrum of scales we often refer to turbulent eddies. A turbulent eddy can be thought of as a local swirling motion whose characteristic dimension is the local turbulence scale. Eddies overlap in space, large ones carrying smaller ones. Thus, turbulent flows are always dissipative. It is also observed that the most important feature of turbulent flows from an engineering point of view is the enhanced diffusivity. Turbulent diffusion greatly enhances the transfer of mass, momentum and energy. Apparent stresses often develop in turbulent flows that are several orders of magnitude larger than in corresponding laminar flows [45].

The nonlinearity of the Navier-Stokes equation leads to interactions between fluctuations of different spectrums and directions. The scale of turbulent flows usually spread all the way from a largest comparable to the width of the flow to a smallest driven by viscous energy dissipation. The process that drives the flows over a wide range of scale is called vortex stretching [5]. Turbulent flow produces and distributes energy if the vortex elements are oriented in a direction in which the mean velocity gradients can stretch them. The wavelengths which are comparable enough to the mean flow width interact most strongly with the mean flow. The larger scale of turbulent motion carries most of energy and responsible for the enhanced diffusivity and increasing stresses.

The process of vorticity generation is not just a creation of the velocity derivatives. It also includes the presence of small scale structure which the inevitable process of vortex stretching tilted and folded because of the cascading process. The strain rate builts up together with limitations on the large scale will leads to the formation of the small scale structure. The definition of small scales above has some consequences.

For example, since the velocity at the large scale is determined by vorticity, the production of vorticity goes back in generating velocity. Since, the velocity is a function of the strain, hence, that production of strain also goes back on the velocity. Therefore, from the physical viewpoint it seems inconvenience to treat the small scales as a kind of passive objects swept by the large scales. Similarly it seems impossible to eliminate the small scales as is done in many theories reducing their reaction back to some eddy viscosity or similar things only [52]. It is also noted that it creates some large scale velocity due to nonlocal relations of small scale vorticity and strain. This and other aspects of nonlocality contradict the idea of cascade in physical space, which is local by definition [53]. According to the above arguments it looks that the energy is transferred not necessarily through a multistep cascade process. Instead, there is also an energy transfer in both directions, whereas the dissipation always occurs in small scales.

In this case, the larger eddies stretch the vortex elements in random way, which comprise of smaller eddies and cascading energy to them. The special interesting feature of a turbulent shear flow is the way large bodies of fluid moving across the flow, also injects smaller scale disturbances to it. The presence of larger eddies near the interface of turbulent region and non turbulent fluid can strongly change the interface. In addition to migrating across the flow, they have a lifetime so long that persist for distances as much as 30 times the width of the flow [5]. Therefore, the turbulent stresses at a certain position will depend on the upstream evolution and cannot be determined uniquely by local strain rate as known in laminar flow.

They are recognised as random fields of vorticity with substantial vortex stretching and a production of enstrophy by inertial nonlinear process dissipated by viscosity. Also, an important process is the production of strain as explained before. Both are the results of accumulation process of the velocity derivatives in turbulent flows, and consist of one of the most important dynamical properties of turbulence. The production of strain is strongly related to the dissipation process of turbulent flows and the amplification of vorticity, is interacted with dissipation. It is known that random potential flows are not turbulence and it is not difficult to observe vorticity dynamics of the three-dimensional turbulent flows, which can be seen from the direct numerical simulations of the Navier-Stokes equations [54]. In fact, the classification of two dimensional chaotic flows with many degrees of freedom as turbulence could be questionable. The main objection is that the two dimensional flows lack the mechanism of vorticity and strain amplification.

The association with the rate of strain tensor which is represented by vortex compressing or the generation of negative enstrophy makes the amplification to the rate of enstrophy generation become positive [55]. However, the non viscous rate of enstrophy generation also comprise of a term representing the interaction between vorticity and the pressure. Stretching of vortex is required to maintain the fluctuating vorticity in a turbulent flow [56]. It is known that the largest deviation from Gaussian distribution is present at the smallest distances between two points or small scale. The velocity derivative is more intermittent than the field of velocity itself. One of the possible reasons for this is in the different nature of nonlinearity at the level of velocity field, i.e. in the Navier-Stokes equations and, may be, in the equation for vorticity. Namely, the nonlinearity in the Navier-Stokes equations contains a potential part and this can be included in the pressure hessian [57]. It is noteworthy to stress that there might not be such reduction of nonlinearity on vorticity scale. Therefore, intermittency can be considered as a product of the nonlinearity.

The mathematical theory is fairly complete in two dimensional case but not in three dimensional or more. This inherent three dimensionality means that there may not be sufficient two dimensional models of the original problem and this is one of the reasons turbulence remains the most unsolved scientific problem of the mathematical physics. The time dependent property of turbulent flow is the main cause of its intractability. Even, many consider that the complexity is beyond the mere introduction of an additional dimension. Turbulence is characterised by random fluctuations thus obviating a deterministic approach to the problem and many people use statistical methods. On the other hand, this aspect is not really a problem from the engineer's view.

While the mean parameter of the nonlinear term in the energy equation is vanishing, the nonlinearity is generating vorticity and strain in physical space as the mean enstrophy and strain are strictly positive. It is reasonable and justified from the physical viewpoint to relate the velocity derivatives with small scales and can be immediately seen that three dimensional turbulent flows have a natural tendency to create small scales. The velocity field and its energy generating in the process of the production of velocity derivatives is the one which is related to the small scales. This process is what can be called as energy transfer from large to small scales in physical space. Indeed, as mentioned earlier, two large nearby eddies can dissipate energy directly by encountering each other on a very small scale [58].

Since there is no precise conclusion in defining the meaning of the term scales, then the meaning of the term cascade which is associated with the spectral energy transfer suffer ambiguity. The common viewpoint is that, the small scales are always related with the velocity derivatives. Thus, it is reasonable to consider this field as the one representing the small scales of turbulent flows. The dissipation is related to the symmetric part of the velocity derivative which represents the rate of strain, while vorticity is related to the anti symmetric part. Meanwhile, the large scales are naturally characterised by the velocity field itself. This is also justified because the sustaining turbulent flows requires energy input into the flow, the power input associated with this force is the velocity field [46]. Even if we had a complete time history of a turbulent flow, we would usually integrate the flow properties of interest over time to extract time averages.

However, the technique of time averaging that lead to statistical correlations in the equations of motion cannot be determined a priori. This is the classical closure problem. In principle, the time dependent, three dimensional Navier-Stokes equation contains all of the physics of a given turbulent flow. That is true from the fact that turbulence is a continuum phenomenon. In fact, it is observed that the smallest scales in turbulent flows are far larger than molecular length scale. Nevertheless, the smallest scales of turbulence are still extremely small. They are generally many orders of magnitude smaller than the largest scales of turbulence, the latter being of the same order of magnitude as the dimension of the object about which the fluid is flowing. Furthermore, the ratio of the smallest to largest scales decreases rapidly as the Reynolds number increases. To make an accurate numerical simulation (a full time dependent three dimensional solution) of a turbulent flow, all physically relevant scales must be resolved.

Nevertheless, it is possible to generate fully resolved solutions at moderate Reynolds numbers through direct numerical simulations (DNS) of the Navier-Stokes equations. Following a goal to reveal complete time history of turbulent flows, only solutions to the full set of the Navier-Stokes equation will be convenient. The accepted solutions require a highly accurate numerical solver and could require the use of subtle transform techniques, not to mention vast computer resources. At this point, the Navier-Stokes equations most possibly contain more than any approximations of the dimensions of attractors and anything which belongs to the reduction into the dynamical systems. This is not clear, since the Navier-Stokes equations are a gradient expansion from the classical kinetic theory. Therefore, in principle, higher order terms may become dominant in regions with large velocity gradients. Even though the Navier-Stokes equations have a limited kinetic foundation [59], they are generally believed to be adequate describing most of the continuum flows. However, the standpoint of continuum mechanics can be taken at the very beginning. In the case of the related problem between the stress and the rate of strain in the fluid flow, the Newtonian fluid is the one in which this relation is linear. There exists large empirical evidence that the Navier-Stokes equations are valid, at least, at all known practical Reynolds numbers, hence continuum mechanics is also eligible. This also covers the possibility that in special conditions, where the strain rate is extremely large, the Newtonian fluids become non-Newtonian, in which the analysis is shifted to variable viscosity [60].

The most well-known qualitative properties turbulent flows are similar and they also generate the idea of qualitative universality of turbulent flows. The concept of qualitative universality is not just an obscured idea. These qualitative features of turbulent flows are universal for all turbulent flows arising in qualitatively many routes and circumstances and generally characterise turbulent flows in a unified view. There are also universal quantitative properties which are specific for a special class of turbulent flows. Many quantitative properties most possibly will widely vary with the range of scales of interest. The properties of the large scales depend on the mechanisms related to the turbulence excitations which are quantitatively not universal, though they are qualitatively universal. It is the small scale turbulence which, since Kolmogorov, is believed to hold some universal properties that are independent of the large scale flow structures [61]. This point of view is not accepted universally.

The issue of the existence of universal properties is one of several debated controversies in the turbulence problems. This includes the meaning of the term universality. For example, one issue discusses the invariance of some properties of a particular turbulent flow at large enough Reynolds numbers. Another issue is concerned with the universality of scaling properties of small scale turbulent flows, which has remained to be one of the most active targets of inquiry. The scaling property of turbulent flows which is derived from the first principle is one of the most popular objectives of fluid dynamics research. Scaling and other phenomenological aspects are extensively reviewed in literatures [62].

The nonlinearity of the Navier-Stokes equations is the most frequently pointed out as the main source. The nonlinearity of the Navier-Stokes equations is obscure, thus not like nonlinear problems that are completely integrable. The famous example is systems which have solitons or solitary waves as a solution. In these systems, the many degrees of freedom are so coupled that they do not show any chaotic and irregular behavior, also they are entirely organised and regular [34]. It is unfortunate that the coherent structures in turbulent flows for example, fall to be treated and viewed in a similar way. The nonlinearity of the Navier-Stokes is also responsible for the difficulties in the closure problem in turbulence modeling which is associated with decomposition, such as the Reynolds decomposition of the velocity into the mean and the fluctuations, or similar decompositions into resolved and unresolved scales related with large eddy simulations (LES) [61]. The main question is the mean field or resolved scales contain seed of the fluctuations or in unresolved scales due to the nonlinearity of the Navier-Stokes equations have to be cleared. A similar problem exists for the advection-diffusion equation representing the dynamics of a passive scalar in some flow fields [63]. But this equation is linear. The problem occurs due to the multiplicative operation of the velocity field, since velocity enters this equation as its coefficients.

It is widely known that the large scale evolution will depends on the fluctuations or unresolved scales in the time and space domain. Therefore, the considered problem cannot be described properly by passive scalar. This means that in turbulent flows, the localised independent relation such as stress-strain relation, can not exist, even though eddy viscosity and eddy diffusivity are considerably implemented as an approximation for describing the reaction back of fluctuations on the mean flow [43]. The fact that the eddy viscosity and eddy diffusivity are flow dependent is just another expression to represent the strong coupling between the large and the small scales [64].

This in turn means that the large scales and the small scales as should be strongly coupled, as indeed is the case. This coupling is in two directions, which means that the small scales cannot be consider as passive to the large scales and the small scales react back in due to the nonlocality. Somehow such relation in the case of a passive object is detected in a turbulent flow [65]. The coupling of large and small scales which is related to decomposition of turbulent flows and in the closure problems occurs frequently. The relation between the fluctuations and the mean flow is not localised in space time, it is a functional [66]. From the mathematical point of view, a process is called local if all the terms in the governing equations are differential. When the equations consist of integral terms, the process is non local. It is already defined that the Navier-Stokes equations are classified as integro differential in the velocity and represent non local processes. The problem is strongly associated to the decompositions, for example, replacing pressure term by a local quantity may not turn the problem into an integrable system. However, the reason for the formation of singularity in finite time in such models is that the scheme in the integrable models is fixed in space, whereas in a real turbulent flow it is oriented randomly in space and time [67]. This means that nonlocality due to pressure is substantial for self sustaining turbulence.

Physically, it is satisfactory to state that the nonlocality is because of the presence of long range forces due to pressure [16]. Since the pressure is nonlocal due to nonlocality of the integral operator, the pressure is then defined in each space point by the velocity in the whole flow field, which is related also to the nonlocality in time. Nonlocality cannot be easily dropped by implementing the curl operator to the Navier-Stokes equations which eliminate the pressure gradient term and producing vorticity equations. The situation is that the vorticity is nonlocal in vorticity equations, since it borrows the strain rate due to the nonlocality of the operator. The whole flow field is defined in each space point by the vorticity and boundary conditions on velocity.

A related aspect is that the acceleration which is a kind of small scale quantity is dominated by pressure gradient [68]. The vorticity equations including enstrophy are nonlocal in vorticity, they consist of the strain rate tensor and the nonlocal interaction exists between vorticity and the strain rate itself. Also, in compressible flows there is no such relatively simple relation between pressure and velocity gradient, but the vorticity-strain relation remains the same [69]. The nonlocality is also observed for the strain rate, dissipation and for the third order quantities. An essential aspect is that the dynamics of nonlocality due to pressure hessian can be viewed as interaction between vorticity and pressure and between strain and pressure. The production of enstrophy describes nonlocal aspects of vortex stretching process.

Transitions from one flow to another with increasing Reynolds number are observed to be a representation of structural evolutions of the mathematical objects called phase flow and attractors through bifurcations [70]. However, weak turbulent flows do not fit easily in this conjecture. Weak turbulent flows contain continuous transition from laminar flow into turbulent as result of the process called entrainment. The processes by which the transition to turbulence are quite diverse, all known quantitative properties of many turbulent flows may appear to be weakly dependent either on the initial conditions or on the history and particular way of their creation, like the flows can be started from rest or from the other flows. Even though, the quantitative properties of turbulent flows could depends on the nature of their transition, it is well known that the qualitative properties of turbulent flows remain the same.

There is variety of processes by which the transition phenomena to turbulent flows is due to the details of instability [71]. Many flows, such as internal flows, boundary layers, jets, shear flows are easily affected by external noise and excitation. There are substantial differences in the instability of turbulent shear flows which consists of free and wall bounded flow, thermal convection, vortex breakdown, surface and internal waves as the major phenomena [72,73,74,75,76]. It is important to note that such differences also exist for the same geometry, which displays interesting variety of transitional behaviour. The unique route could depend on initial conditions, external disturbances, external force, time history and other details. This difference is practically observed at the initial state of the linear instability, where the nonlinear stages are less sensitive to such details.

It is quite common to confront the traditional statistical and the deterministic or structural approaches in turbulence research. However, contrasting the terms deterministic and random has lost most of its meaning or at least become blurred with the developments in deterministic chaos. It is noteworthy to mention that simple systems governed by a deterministic nonlinear equations, shows irregular, random and stochastic behaviour [77]. The problem of turbulence was known long before the construction of chaos theories. Since Leray, there is no universal agreement whether turbulence is a breakdown of the Navier-Stokes equations [78].

In turn, one of the outstanding problems in mathematical physics is whether the Navier-Stokes equations at large Reynolds numbers develop a genuine singularity in finite time. Note that there is some analytical and numerical evidence that, at least for Euler equations, this may be true [79,80]. Also, it seems a justified view that the presence of singularities will develop topological defect and dissipation for the Navier-Stokes equations. Their existence is influenced at the dissipation scales and is perhaps the source of small scale intermittency [78]. Such reaction back is reasonable due to the direct coupling between large and small scales. Near singular objects related with non-integer values of the energy spectrum scaling exponents are investigated to be closely associated with some structures and with intermittency of turbulent flows [81,82]. In most cases, the near singular objects may be among the source of intermittency. The problem with two dimensional turbulence is that everything is found regular, but there is still intermittency and near Gaussian behavior. However, non Gaussian property is strong at the level of velocity derivatives of a second order [83]. Therefore, the possibility of singularity formations in three dimensions is not always the cause for intermittency in three-dimensional turbulent flows.

The important point is that investigating the behavior of a simplified equation will not solve the problem. Even any particular solutions from analytical methods may have only little contribution to the understanding of the basic properties of turbulent flows. Consequently, nothing less than by understanding the global behaviour of solutions of the Navier-Stokes equations would seem to be convenient to explain the phenomenon of turbulence [54]. The problem is to solve the constitutive equations with subject to the initial and boundary conditions. The task is difficult because our ability to tackle nonlinear problem i.e. very high dimension and complicated structure of the underlying attractors is still in the early development.

As a summary, turbulence consists of many physical natures which seem to be difficult to solve. Turbulence also seems to have qualitative universal feature for many cases. The understanding is still not clear whether this feature is strongly related to the variety of the large scale dynamics of turbulent flows. Note that this justification mostly comes from experimental evident, because the development of numerical and mathematical analysis of the Navier-Stokes equations is still not sufficient to tackle phenomenological problems. On the theoretical side, once the problem of singularity is settled, mathematical analysis will at one big step forward to reveal the secret of turbulence orchestrated by the Navier-Stokes equations.

Chapter 3

Contributions to the Theory of Solutions: The Physical and Mathematical Aspects of the Incompressible Navier-Stokes Equations

As they are well defined, the incompressible Navier--Stokes equations with continuity equations are presumed to embody all of the physics inherent in all fluid flow phenomena, or more specifically, laminar and turbulent viscous incompressible fluid flows. It is known that the Navier-Stokes equations are typically a problem of non-integrable systems, whose any global unique solutions may not exist.

The existence of time periodic solution to the Navier-Stokes equations is proved in the whole space [84]. Then, more generally, strong solvability of the Navier-Stokes equations is investigated [85]. It is proved that there exists essentially only one maximal strong solution and that various concepts of generalised solutions coincide. Some criteria on certain components of gradient velocity are given to ensure global smoothness in time [14,86]. Considerable effort is spent to reduce the analysis to make it more tractable, like partial regularity of the nonstationary Navier-Stokes equations in $\Omega\times[0,T]$ where the regularity of suitable weak solutions is proven for large $|x|$ [87]. It is also mentioned that their result also holds near the boundary. The more general regularity concept with simplified problem is investigated in [88] which states that w satisfies either $w\in L^{\infty}\left(\mathbb{R}^2\times(0,T)\right)$ or $\vec{\nabla}w\in L^{p}\left(0,T;L^{q}\left(\mathbb{R}^3\right)\right)$ with $1/p+3/2q=1/2$ and $q\geq 3$ for some $T>0$ then u is regular on $[0,T]$. Similar investigation is performed for thin three-dimensional flows [89].

3.1 Triviality in Boundary Value Problems

Global in time continuation still remains the unsolved problem in mathematical fluid mechanics [89,90]. The question arises as if such singularities exist, they might be related to turbulence by invoking that we have global smooth solution for two-dimensional flows, and turbulence is three-dimensional phenomena. This hypothesis, however, has serious difficulties as the observed phenomenon is so far bounded in nature.

Therefore, the argument that turbulent solutions should have no singularities is supported in this work based on the triviality of the solution for simple energy equation. Analysis of global trivial solutions is important from mathematical and physical aspects, it has wide application due to its correlation with many areas where some hierarchical solutions are needed to be arranged [15,26,38]. Classical procedure of vector identities is implemented for producing trivial solutions. Violation from trivial solutions is also investigated. Investigation of nontrivial solutions is related to the rate of energy generated or destroyed. The assumed nontrivial solution from the Navier-Stokes equations is performed by the utilisation of the vorticity equations which is related to the onset of turbulence due to energy accumulations.

3.1.1 Boundary Value Problems

Let Ξ be the region of interest as described in fig. 3.1. It is supposed that the associated problem is a connected, bounded region in three-dimensional domain. Moreover, let $\Xi_i, i=1,...,m$, be sub regions characterised by simple boundaries and $\Xi_{ai}, i=1,...,n$ be the sub regions where boundaries are not simple. This means that the considered boundary $\partial\Xi_i$ and $\partial\Xi_{ai}$ are defined as regular and irregular surfaces respectively.

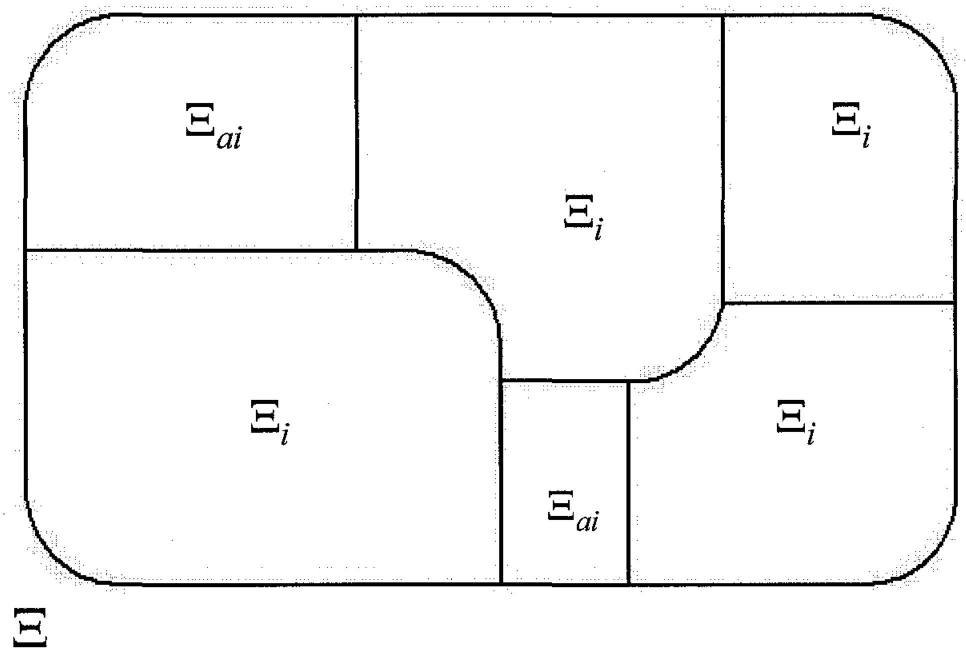


Fig. 3.1 Region and sub regions of interest

Thus, the boundary-value problem is determined as follows; given density $\rho > 0$ such that velocity $\vec{V}$ is real vector field consists of $<u,v,w>$ components and p is real scalar field defined in every region $\Xi_i, i=1,...,m$, and $\Xi_{ai}, i=1,...,n$ which fulfill,

$$\frac{\partial \vec{V}}{\partial t} + \vec{V} \cdot \vec{\nabla} \vec{V} = -\frac{1}{\rho} \vec{\nabla} p + \upsilon \vec{\nabla}^2 \vec{V} \tag{3.1a}$$

$$\vec{\nabla} \cdot \vec{V} = 0 \tag{3.1b}$$

with boundary and initial conditions $\vec{V} = \vec{V}(x,y,z,0)$ on $\partial\Xi_1$ or $\partial\Xi_{a1}$. The previous problem also denotes kinematic viscosity $\upsilon > 0$, $\vec{\nabla} = <\partial/\partial x, \partial/\partial y, \partial/\partial z>$ and $\vec{n}$ is the unit vector normal to the surfaces S parallel to the velocity. It will be demonstrated that if the nonzero divergence condition in (3.1a) due to the production term in the control volume is implemented, the boundary value problem does not have any trivial solutions. It is very important consequence since the nonzero divergence in the continuity equation has wide applications, for example, combustion problems.

It is noted that the previous boundaries are characterised by the Cartesian coordinates and only applied to few simple applications. For this reason it is important to prove that the associated problem can be extended to general coordinates for non-simple boundaries. The general coordinates are also defined in $\Xi_i, i=1,...,m$ and

$\Xi_{ai}, i = 1,\ldots,n$ such that general solution is mapped to the general coordinates in the same regions. The mathematical formulation of the associated general coordinates is,

$$\xi = \xi(x, y, z, t) \tag{3.2a}$$

$$\eta = \eta(x, y, z, t) \tag{3.2b}$$

$$\varsigma = \varsigma(x, y, z, t) \tag{3.2c}$$

$$\tau = \tau(t) \tag{3.2d}$$

such that ξ, η, ς and τ are continuous in $\Xi_i, i = 1,\ldots,m$ and $\Xi_{ai}, i = 1,\ldots,n$. It is also important to note that the boundary conditions considered in (3.1) are also applied to the general coordinates.

3.1.2 The Existence of Trivial Solutions

In this section, some important properties of the solution of (3.1a – b) are investigated in order to prove the triviality of the solutions with respect to the boundary value considered. The energy rate is defined as a product of static pressure p and flow rate $\vec{V}$ across control surface. Direction of velocity is parallel to the unit normal control surface $\vec{n}$. It is supposed that the region Ξ be the associated problem and consists of i parts of region with simple and non simple boundaries. Therefore, the assumed zero energy rate, E can be written as,

$$\frac{\partial E}{\partial t} = pA\vec{V} + \tfrac{1}{2}\rho\vec{V}^2 A\vec{V} = 0 \tag{3.3}$$

Consequently, the divergence theorem can be applied to the whole region of interest as,

$$\oiint_S \left(p\vec{V}\cdot\vec{n} + \tfrac{1}{2}\rho\vec{V}^2\vec{V}\cdot\vec{n}\right)dS = \iiint_\Xi \vec{\nabla}\cdot\left(p\vec{V} + \tfrac{1}{2}\rho\vec{V}^3\right)d\Xi = 0 \tag{3.4a}$$

By using vector identity, $\vec{\nabla}\cdot\left(f\vec{F}\right) = f\vec{\nabla}\cdot\vec{F} + \vec{F}\cdot\vec{\nabla}f$, hence, it is identified,

$$\oiint_S \left(p\vec{V}\cdot\vec{n} + \tfrac{1}{2}\rho\vec{V}^2\vec{V}\cdot\vec{n}\right)dS = \iiint_\Xi \left(p\vec{\nabla}\cdot\vec{V} + \vec{V}\cdot\vec{\nabla}p + \frac{1}{2}\rho\vec{V}^2\vec{\nabla}\cdot\vec{V} + \frac{1}{2}\rho\vec{V}\cdot\vec{\nabla}\vec{V}^2\right)d\Xi = 0 \tag{3.4b}$$

Since $\oiint_S \left(p\vec{V}\cdot\vec{n} + \tfrac{1}{2}\rho\vec{V}^2\vec{V}\cdot\vec{n}\right)dS = 0$, then

$$\sum_{i=1}^{m}\iiint_{\Xi}\left(p\vec{\nabla}\cdot\vec{V}+\vec{V}\cdot\vec{\nabla}p+\frac{1}{2}\rho\vec{V}^{2}\vec{\nabla}\cdot\vec{V}+\frac{1}{2}\rho\vec{V}\cdot\vec{\nabla}\vec{V}^{2}\right)d\Xi+$$
$$\sum_{i=1}^{n}\iiint_{\Xi}\left(p\vec{\nabla}\cdot\vec{V}+\vec{V}\cdot\vec{\nabla}p+\frac{1}{2}\rho\vec{V}^{2}\vec{\nabla}\cdot\vec{V}+\frac{1}{2}\rho\vec{V}\cdot\vec{\nabla}\vec{V}^{2}\right)d\Xi=0 \qquad (3.4c)$$

Suppose that the rate of energy is zero everywhere, then, equation (3.4c) is automatically satisfied. The first and third terms of energy rate are always zero according to continuity. Therefore, since p and $\vec{V}$ are nonzero, then $\vec{\nabla}p$ and $\vec{\nabla}\vec{V}$ must be zero. Thus, the following trivial solution is defined,

$$p = \text{constant and } \vec{V} = \text{constant} \qquad (3.4d)$$

It is noted that equation (3.4d) will also satisfy the continuity and incompressible Navier-Stokes equations. This is one of the interpretations of equation (3.4c).

However, it is interesting to consider more general conditions in which (3.4c) will be satisfied. Consider the function not zero for whole infinite boundaries,

$$F=\int_{m}\iiint_{\Xi_1}\left(p\vec{\nabla}\cdot\vec{V}+\vec{V}\cdot\vec{\nabla}p+\frac{1}{2}\rho\vec{V}^{2}\vec{\nabla}\cdot\vec{V}+\frac{1}{2}\rho\vec{V}\cdot\vec{\nabla}\vec{V}^{2}\right)d\Xi_1 dm+$$
$$\int_{n}\iiint_{\Xi_{a1}}\left(p\vec{\nabla}\cdot\vec{V}+\vec{V}\cdot\vec{\nabla}p+\frac{1}{2}\rho\vec{V}^{2}\vec{\nabla}\cdot\vec{V}+\frac{1}{2}\rho\vec{V}\cdot\vec{\nabla}\vec{V}^{2}\right)d\Xi_{a1} dn \qquad (3.5a)$$

It is possible that $n << m$, and without losing a generality take $n=1$, such that by dividing procedure the following expression is produced from equation (3.5a),

$$z=g(n)+\int_{m}h(m)z(m)dm \qquad (3.5b)$$

where,

$$g(n)=\iiint_{\Xi_{a1}}\left(p\vec{\nabla}\cdot\vec{V}+\vec{V}\cdot\vec{\nabla}p+\frac{1}{2}\rho\vec{V}^{2}\vec{\nabla}\cdot\vec{V}+\frac{1}{2}\rho\vec{V}\cdot\vec{\nabla}\vec{V}^{2}\right)d\Xi_{a1}$$
$$h(m)z(m)=\iiint_{\Xi_1}\left(p\vec{\nabla}\cdot\vec{V}+\vec{V}\cdot\vec{\nabla}p+\frac{1}{2}\rho\vec{V}^{2}\vec{\nabla}\cdot\vec{V}+\frac{1}{2}\rho\vec{V}\cdot\vec{\nabla}\vec{V}^{2}\right)d\Xi_1 .$$

Note that z in the left hand side is same as right hand side. By taking $j=\int_{m}h(m)z(m)dm$, where $z\le j$, and performing differentiation of j, the following inequality is produced,

$$j'=h(m)z(m)\le h(m)j(m) \qquad (3.5c)$$

By multiplying (3.5c) by $\exp\left(-\int h(m)dm\right)$, and applying the identity [91] to obtain,

$$j'\exp\left(-\int h(m)dm\right)-h(m)\,j(m)\exp\left(-\int h(m)dm\right)\leq 0$$

Therefore, the following result is yielded by integration,

$$j\leq c\exp\left(\int h(m)dm\right) \tag{3.5d}$$

with c being an arbitrary constant. Since $j\leq F$, it is proved that zero value for F is very special case for condition that energy rate is not zero everywhere. Moreover, the consideration can be changed to $j\leq z$ and $F\leq j$, by the same procedure in (3.5a - 3.5d), to obtain the inequality $c\exp\left(\int h(m)dm\right)\leq j$. It can be concluded that the special condition still holds and case for zero energy rate everywhere is more plausible. Therefore, the above derivation can be stated in the following lemma,

Lemma 1: *There exist p and V as solutions to the continuity and incompressible Navier-Stokes equations such that the rate of energy in the whole domain of $\Xi_i, i=1,\dots,m$ and $\Xi_{ai}, i=1,\dots,n$ with respect to the initial and boundary conditions $\vec{V}=\vec{V}(x,y,z,0)$ on $\partial\Xi_1$ or $\partial\Xi_{a1}$, is equal to zero. The solutions then are,*

$$p = \textit{constant and } \vec{V} = \textit{constant}$$

in Ξ_i and Ξ_{ai}.

It is obvious that the above trivial solution will only be valid in special cases. The violation of the condition described in lemma 1 is strongly related to the Navier-Stokes equations which can be correlated to the onset of turbulence as explained in section 3.1.4. In this case, the analyticity is implemented in order to generalize solutions to the limit of or near to triviality in some sharing regions. Hence, another interesting result is produced,

Proposition 1: *Any nontrivial solutions are analytic in regions Ξ_i* and Ξ_{ai}.

Furthermore, in the intersection region such that $\Xi_i\cap\Xi_{ai}$ is strictly held, function $f(x)$ is considered. Suppose that $f(x)$ is at least twice differentiable and convergence to the some constant value $\dot{B}$ in certain location which is related to lemma 1.

According to the property of analytic function, it is reasonable to assume that if $f(x)$ is convergent in some point y which is close enough to x then some value B' will be generated such that $B' \approx B$. Therefore, for p and $\vec{V}$ with associated boundary conditions, it is proper to consider sub region Ψ in $\Xi_i \cap \Xi_{ai}$ such that $x \le \Psi \le y$ where $\Psi \to x \in \Xi_i$ and $\Psi \to y \in \Xi_{ai}$. Then, according to the analyticity property of p and $\vec{V}$, it is reasonable to conclude that p and $\vec{V}$ in Ψ are equal to B' and C' where $B' \approx B$ and $C' \approx C$ in $\Xi_i, i = 1,...,m$ and $\Xi_{ai}, i = 1,...,n$. Therewith, the following statement is produced,

Lemma 2: *Given any nontrivial solutions such that $p > 0$ and $\|\vec{V}\| > 0$, can be applied to Λ, where Λ is a boundary of Ξ_i and also becomes at least one sub region or part of Ξ_{ai}, where p and $\vec{V}$ are trivial. The corresponding solution can be interchanged at the boundary such that there exist*

$$p \approx constant \text{ and } \vec{V} \approx constant$$

in Λ.

It is interesting to note that shock wave and laminar-turbulence interface problems might be described by this condition since they are considered as a discontinuity jump between two regions separated by boundary conditions.

3.1.3 Triviality in General Coordinates

First, it is observed that equation (3.1) in general coordinate (3.2) will satisfy the energy conditions (3.4) in the whole domain. Then, by utilising lemma 1, the following result is produced,

$$p = \text{constant and } \vec{V} = \text{constant}$$

in $\Xi_i, i = 1,...,m$ and $\Xi_{ai}, i = 1,...,n$. Then, proposition above is used to observe that p is analytic with its derivatives in $\Xi_i, i = 1,...,m$ and $\Xi_{ai}, i = 1,...,n$. Sub regions $1 \le k \le m$ in

$\Xi_i, i=1,...,m$ and $1 \le l \le m$ in $\Xi_{ai}, i=1,...,n$ are now considered. There exist some regions that $1 \le k \le m$ and $1 \le l \le m$ are intersect such that $\Xi_i \cap \Xi_{ai}$. If the solution is nontrivial in $\Xi_i, i=1,...,m$ or $\Xi_{ai}, i=1,...,n$, then it is analytic due to proposition 1. As a consequence, the following result is also obtained at the boundary,

$$p \approx \text{constant and } \vec{V} \approx \text{constant}$$

Therefore, the following theorem is just already proved,

Theorem 1: *Any solutions for boundary value problems of the continuity and incompressible Navier-Stokes equations that satisfy the condition of zero rate energy is trivial, i.e.* $p =$ *constant and* $\vec{V} =$ *constant.*

3.1.4 A Possible Route to Turbulence

It is more reliable to study the fluid motion using vorticity [29,41]. Taking curl operation to the Navier-Stokes equations, the following vorticity equations are obtained,

$$\frac{\partial \vec{\omega}}{\partial t} + \vec{V} \cdot \vec{\nabla} \vec{\omega} = \vec{\omega} \cdot \vec{\nabla} \vec{V} + \upsilon \vec{\nabla}^2 \vec{\omega} \tag{3.6}$$

where $\vec{\omega} = \vec{\nabla} \times \vec{V}$ and $\vec{\omega} = <\omega_x, \omega_y, \omega_z>$. Note that the pressure term in (3.1a) is vanished by the curl procedure and equations (3.6) also satisfy continuity equation for incompressible flow, $\vec{\nabla} \cdot \vec{V} = 0$. It is supposed that there exists a potential function Φ such that the velocity vector can be expressed as,

$$\vec{V} = \vec{\nabla} \times \Phi \tag{3.7}$$

Therefore, the vorticity can also be expressed by the potential function Φ as,

$$\omega = \vec{\nabla} \times \left(\vec{\nabla} \times \Phi \right) \tag{3.8}$$

Note that equation (3.6) has some solutions for certain boundary and initial conditions. Therefore, any solutions of (3.6) will produce velocity field in the following form,

$$\vec{V} = \frac{\partial \Phi}{\partial x_j} - \frac{\partial \Phi}{\partial x_i} \tag{3.9}$$

Hence, by substituting (3.9) into the Navier-Stokes equations, then the solution for p is non-trivial, thus, the above derivation can be concluded as in the following,

Proposition 2: *Boundary value problems of nonstationary three-dimensional incompressible flows admit non trivial solutions*

Note that the problem considered here strictly obeys the Navier-Stokes equations, so violation from condition explained before, i.e. $p = \text{constant}$ and $\vec{V} = \text{constant}$ are very possible. Suppose that due to relation (3.3), the energy rate is produced if the condition violated, it is plausible that the excess energy will be distributed to the whole domain, then the observed parameters will also deviate from trivial conditions. Furthermore, the case considered here strictly admits continuity in the form of velocity divergence, meaning that if in some cases velocity divergence is not zero, triviality in p and $\vec{V}$ will be more difficult to obtain.

Corollary 1: *Any non trivial solution for boundary value problems of the continuity and incompressible Navier-Stokes equations is a possible onset of turbulence.*

which is also stated by Adomian [92]

3.2 The Mathematical Theory

Apart from the mathematical analysis, it is known that the solution of the Navier-Stokes equations on the corresponding domain with periodic boundary conditions has global regularity, as long as there is control on the size of initial data and the forcing term. Also, the Navier-Stokes equations are modified in a lengthy work of [93] to find the interior regularity and to ensure the uniqueness of the solutions. However, it is possible to generate the existence theorem from explicit solutions like numerical methods [94] to provide, by strict solution, a rigorous a posteriori analysis of the existence of the steady solutions.

Therefore, it is clear that although it is promising to overcome the problem of nonlinear differential equations by finding class of exact solution [95], it is important to give the foundations of the analytical solutions to explore their global properties, like one might find possible local singularity for this particular class of the solutions [96]. Therefore, this section provides analysis of analytical solutions which are detailed in section 4.4. Analysis is carried out in vorticity equations rather than in the Navier-Stokes equations by considering that solutions will fulfill certain conditions that satisfy the Navier-Stokes equations [41]. Additional assumption for the pressure condition is not necessary since it will vanish through the curl procedure. The obtained solution is then substituted back to the original Navier-Stokes equations and the pressure relation is also obtained. In this work, a potential function is proposed to form the special classes of solution.

3.2.1 Triviality in $\vec{V} = \vec{\nabla} \times \Phi$ and $\vec{V} = \vec{\nabla}\Phi + \vec{\nabla} \times \Phi$ Classes of Solutions

Consider equations (3.6), (3.7) and (3.8). Thus, the vorticity can be defined explicitly as,

$$\omega_x = \Phi_{xy} - \Phi_{yy} - \Phi_{zz} + \Phi_{xz} \tag{3.10a}$$

$$\omega_y = \Phi_{yz} - \Phi_{zz} - \Phi_{xx} + \Phi_{xy} \tag{3.10b}$$

$$\omega_z = \Phi_{xz} - \Phi_{xx} - \Phi_{yy} + \Phi_{yz} \tag{3.10c}$$

Substitute equation (3.10a – c) into the vorticity equations (3.6) will yield a system of equations,

In x direction;

$$\begin{aligned}
&\frac{\partial\{\Phi_{xy} - \Phi_{yy} - \Phi_{zz} + \Phi_{xz}\}}{\partial t} + \{\Phi_y - \Phi_z\}\frac{\partial\{\Phi_{xy} - \Phi_{yy} - \Phi_{zz} + \Phi_{xz}\}}{\partial x} + \{\Phi_z - \Phi_x\}\frac{\partial\{\Phi_{xy} - \Phi_{yy} - \Phi_{zz} + \Phi_{xz}\}}{\partial y} \\
&+ \{\Phi_x - \Phi_y\}\frac{\partial\{\Phi_{xy} - \Phi_{yy} - \Phi_{zz} + \Phi_{xz}\}}{\partial z} = \{\Phi_{xy} - \Phi_{yy} - \Phi_{zz} + \Phi_{xz}\}\frac{\partial\{\Phi_y - \Phi_z\}}{\partial x} + \{\Phi_{yz} - \Phi_{zz} - \Phi_{xx} + \Phi_{xy}\}\frac{\partial\{\Phi_y - \Phi_z\}}{\partial y} \\
&+ \{\Phi_{xz} - \Phi_{xx} - \Phi_{yy} + \Phi_{yz}\}\frac{\partial\{\Phi_y - \Phi_z\}}{\partial z} + \upsilon\frac{\partial^2\{\Phi_{xy} - \Phi_{yy} - \Phi_{zz} + \Phi_{xz}\}}{\partial x^2} + \upsilon\frac{\partial^2\{\Phi_{xy} - \Phi_{yy} - \Phi_{zz} + \Phi_{xz}\}}{\partial y^2} \\
&+ \upsilon\frac{\partial^2\{\Phi_{xy} - \Phi_{yy} - \Phi_{zz} + \Phi_{xz}\}}{\partial z^2}
\end{aligned} \tag{3.11a}$$

In y direction;

$$\frac{\partial\{\Phi_{yz}-\Phi_{zz}-\Phi_{xx}+\Phi_{xy}\}}{\partial t}+\{\Phi_y-\Phi_z\}\frac{\partial\{\Phi_{yz}-\Phi_{zz}-\Phi_{xx}+\Phi_{xy}\}}{\partial x}+\{\Phi_z-\Phi_x\}\frac{\partial\{\Phi_{yz}-\Phi_{zz}-\Phi_{xx}+\Phi_{xy}\}}{\partial y}$$
$$+\{\Phi_x-\Phi_y\}\frac{\partial\{\Phi_{yz}-\Phi_{zz}-\Phi_{xx}+\Phi_{xy}\}}{\partial z}=\{\Phi_{xy}-\Phi_{yy}-\Phi_{zz}+\Phi_{xz}\}\frac{\partial\{\Phi_z-\Phi_x\}}{\partial x}+\{\Phi_{yz}-\Phi_{zz}-\Phi_{xx}+\Phi_{xy}\}\frac{\partial\{\Phi_z-\Phi_x\}}{\partial y}$$
$$+\{\Phi_{xz}-\Phi_{xx}-\Phi_{yy}+\Phi_{yz}\}\frac{\partial\{\Phi_z-\Phi_x\}}{\partial z}+\upsilon\frac{\partial^2\{\Phi_{yz}-\Phi_{zz}-\Phi_{xx}+\Phi_{xy}\}}{\partial x^2}+\upsilon\frac{\partial^2\{\Phi_{yz}-\Phi_{zz}-\Phi_{xx}+\Phi_{xy}\}}{\partial y^2}$$
$$+\upsilon\frac{\partial^2\{\Phi_{yz}-\Phi_{zz}-\Phi_{xx}+\Phi_{xy}\}}{\partial z^2} \tag{3.11b}$$

In z direction;

$$\frac{\partial\{\Phi_{xz}-\Phi_{xx}-\Phi_{yy}+\Phi_{yz}\}}{\partial t}+\{\Phi_y-\Phi_z\}\frac{\partial\{\Phi_{xz}-\Phi_{xx}-\Phi_{yy}+\Phi_{yz}\}}{\partial x}+\{\Phi_z-\Phi_x\}\frac{\partial\{\Phi_{xz}-\Phi_{xx}-\Phi_{yy}+\Phi_{yz}\}}{\partial y}$$
$$+\{\Phi_x-\Phi_y\}\frac{\partial\{\Phi_{xz}-\Phi_{xx}-\Phi_{yy}+\Phi_{yz}\}}{\partial z}=\{\Phi_{xy}-\Phi_{yy}-\Phi_{zz}+\Phi_{xz}\}\frac{\partial\{\Phi_x-\Phi_y\}}{\partial x}+\{\Phi_{yz}-\Phi_{zz}-\Phi_{xx}+\Phi_{xy}\}\frac{\partial\{\Phi_x-\Phi_y\}}{\partial y}$$
$$+\{\Phi_{xz}-\Phi_{xx}-\Phi_{yy}+\Phi_{yz}\}\frac{\partial\{\Phi_x-\Phi_y\}}{\partial z}+\upsilon\frac{\partial^2\{\Phi_{xz}-\Phi_{xx}-\Phi_{yy}+\Phi_{yz}\}}{\partial x^2}+\upsilon\frac{\partial^2\{\Phi_{xz}-\Phi_{xx}-\Phi_{yy}+\Phi_{yz}\}}{\partial y^2}$$
$$+\upsilon\frac{\partial^2\{\Phi_{xz}-\Phi_{xx}-\Phi_{yy}+\Phi_{yz}\}}{\partial z^2} \tag{3.11c}$$

By taking sum of equation (3.11a – c), the following linear equation is produced

$$\frac{\partial\Gamma}{\partial t}+\Lambda\vec{\nabla}\Gamma=\upsilon\vec{\nabla}^2\Gamma \tag{3.12}$$

with $\Gamma=2\{\Phi_{xy}+\Phi_{xz}+\Phi_{yz}-\Phi_{xx}-\Phi_{yy}-\Phi_{zz}\}$, $\Lambda_1=\{\Phi_y-\Phi_z\}$, $\Lambda_2=\{\Phi_z-\Phi_x\}$, $\Lambda_3=\{\Phi_x-\Phi_y\}$. Therefore, the corresponding problem falls into a category of linear parabolic (for Γ) and elliptic (for Φ) differential equations. Since the assumption of regular boundary is held, construction of weak solution in L^2 and strong solution in L^p can be developed more easily as explained in the next section.

However, a more general solution can be developed using,

$$\vec{V}=\vec{\nabla}\Phi+\vec{\nabla}\times\Phi \tag{3.13}$$

Similar expression in (3.12) can be obtained by the same procedure as above with the additional terms resulting from the assumption of the vortex stretch added to the right hand side of (3.11), for x direction :

$$\left\{\Phi_{xy}-\Phi_{yy}-\Phi_{zz}+\Phi_{xz}\right\}\frac{\partial\left\{\Phi_x+\Phi_y-\Phi_z\right\}}{\partial x}+\left\{\Phi_{yz}-\Phi_{zz}-\Phi_{xx}+\Phi_{xy}\right\}\frac{\partial\left\{\Phi_x+\Phi_y-\Phi_z\right\}}{\partial y} +\left\{\Phi_{xz}-\Phi_{xx}-\Phi_{yy}+\Phi_{yz}\right\}\frac{\partial\left\{\Phi_x+\Phi_y-\Phi_z\right\}}{\partial z} \quad (3.14a)$$

for y direction :

$$\left\{\Phi_{xy}-\Phi_{yy}-\Phi_{zz}+\Phi_{xz}\right\}\frac{\partial\left\{\Phi_y+\Phi_z-\Phi_x\right\}}{\partial x}+\left\{\Phi_{yz}-\Phi_{zz}-\Phi_{xx}+\Phi_{xy}\right\}\frac{\partial\left\{\Phi_y+\Phi_z-\Phi_x\right\}}{\partial y} +\left\{\Phi_{xz}-\Phi_{xx}-\Phi_{yy}+\Phi_{yz}\right\}\frac{\partial\left\{\Phi_y+\Phi_z-\Phi_x\right\}}{\partial z} \quad (3.14b)$$

for z direction :

$$\left\{\Phi_{xy}-\Phi_{yy}-\Phi_{zz}+\Phi_{xz}\right\}\frac{\partial\left\{\Phi_z+\Phi_x-\Phi_y\right\}}{\partial x}+\left\{\Phi_{yz}-\Phi_{zz}-\Phi_{xx}+\Phi_{xy}\right\}\frac{\partial\left\{\Phi_z+\Phi_x-\Phi_y\right\}}{\partial y} +\left\{\Phi_{xz}-\Phi_{xx}-\Phi_{yy}+\Phi_{yz}\right\}\frac{\partial\left\{\Phi_z+\Phi_x-\Phi_y\right\}}{\partial z} \quad (3.14c)$$

Thus, according to the differentiation rule, by assuming that the above equation is equal to,

$$\left\{\Phi_{xxy}-\Phi_{xyy}-\Phi_{xzz}+\Phi_{xxz}\right\}\left\{\Phi_x+\Phi_y-\Phi_z\right\}+\left\{\Phi_{yyz}-\Phi_{yzz}-\Phi_{xxy}+\Phi_{xyy}\right\}\left\{\Phi_x+\Phi_y-\Phi_z\right\} +\left\{\Phi_{xzz}-\Phi_{xxz}-\Phi_{yyz}+\Phi_{yzz}\right\}\left\{\Phi_x+\Phi_y-\Phi_z\right\} \quad (3.15)$$

for all direction, so zero result will be observed. Then, because of (3.13), components of equation (3.12) are redefined as $\Gamma=2\left\{\Phi_{xy}+\Phi_{xz}+\Phi_{yz}\right\}$, $\Lambda_1=\left\{\Phi_x+\Phi_y-\Phi_z\right\}$, $\Lambda_2=\left\{\Phi_y+\Phi_z-\Phi_x\right\}$ and $\Lambda_3=\left\{\Phi_z+\Phi_x-\Phi_y\right\}$.

Equation (3.12) can be transformed further by taking $x=\ln\Lambda_1$, $y=\ln\Lambda_2$, $z=\ln\Lambda_3$, to give,

$$\frac{\partial\Gamma}{\partial t}+\Lambda_i^2\frac{\partial\Gamma}{\partial\Lambda_i}=\upsilon\Lambda_i^2\frac{\partial^2\Gamma}{\partial\Lambda_i^2} \quad (3.16)$$

which becomes a linear parabolic equation with respect to Λ_i, so that the initial value problems of (3.16) will have generalised unique solution [29]. Hence, if $\Gamma(\Lambda_i)$ is held, then $\Gamma(\Lambda_i)$ will admit general classical solution and have global regularity for weak solution in L^2 and strong solution in L^p with the assumption of regular boundary [97]. With this result, Γ can be rewritten in terms of the potential function, Φ as,

$$\Gamma=2\left\{\Phi_{xy}+\Phi_{xz}+\Phi_{yz}\right\}=f\left(\Lambda_i\right)=f\left(\Phi_i\right) \quad (3.17)$$

and may be investigated by nonlinear analysis which depends on the solution of $\Gamma(\Lambda_i)$. However, it is interesting to note that trivial form of linear differential equations can be generated from (3.17). Transforming back $f(\Lambda_i)$ in their original form (x,y,z,t) redefines (3.17) as,

$$\Gamma = 2\{\Phi_{xy} + \Phi_{xz} + \Phi_{yz}\} = q(x,y,z,t) \tag{3.18}$$

Note that previous analysis shows that $q(x,y,z,t)$ satisfies global regularity in (3.16) and that equation (3.18) falls into the category of nonhomogenous linear hyperbolic equations.

3.2.2 Theory of Solutions

This section is concentrated on the existence and uniqueness of the classical, weak and strong solutions. The analysis is based on the maximum principle of the linear parabolic equation in (3.16), which is implemented to determine L^∞ norm estimate and comparison principle, which will be applied in theorems 2 to 8 [97].

Theorem 2: *Let* $\Lambda_j \geq 0$ *and bounded in* Ω, $\Gamma \in C^2$ *satisfy equation (3.16). Then,*

$$\sup_{\Omega} \Gamma(\Lambda_j,t) \leq \sup_{\partial\Omega} \Gamma_+(\Lambda_j,t)$$

Proof: Consider the existence of point (Λ_j^0,t^0) at $\partial\Omega$ such that,

$$\Gamma(\Lambda_j^0,t^0) = \sup_{\partial\Omega} \Gamma(\Lambda_j,t) > 0 \tag{3.19a}$$

The maximum principle asserts the following condition,

$$\frac{\partial\Gamma(\Lambda_j^0,t^0)}{\partial t} \geq 0,\ \vec{\nabla}\Gamma(\Lambda_j^0,t^0) = 0,\ \vec{\nabla}^2\Gamma(\Lambda_j^0,t^0) \leq 0 \tag{3.19b}$$

Then equation (3.16) will result in,

$$\frac{\partial\Gamma(\Lambda_j^0,t^0)}{\partial t} + \Lambda_j^2\vec{\nabla}\Gamma(\Lambda_j^0,t^0) - \upsilon\Lambda_j^2\vec{\nabla}^2\Gamma(\Lambda_j^0,t^0) \geq 0 \tag{3.19c}$$

and (3.19a) is valid. Let $g = e^{\beta t}$ with $\beta \geq 0$ and $g \in C^2$, substituting into (3.16) will result in,

$$L\Gamma = \frac{\partial g}{\partial t} = \beta e^{\beta t} \geq 0$$

Then, for any constant $\varepsilon > 0$,

$$L(\Gamma + \varepsilon g) = L\Gamma + \varepsilon Lg \geq 0 \tag{3.19d}$$

According to the above equation and (3.19a), the following inequality is produced,

$$\sup_{\Omega}\left[\Gamma(\Lambda_j,t)+\varepsilon g(\Lambda_j,t)\right]\le\sup_{\partial\Omega}\left[\Gamma(\Lambda_j,t)+\varepsilon g(\Lambda_j,t)\right]_+ \tag{3.19e}$$

Let $\varepsilon\to 0$, thus theorem 1 is proved.

Therefore, an additional result can also be concluded as below,

Theorem 3: *Suppose that $\Lambda_j \ge 0$ and bounded in Ω, $\Gamma_1, \Gamma_2 \in C^2$ satisfies $L\Gamma_1 \le L\Gamma_2$ in Ω with $\Gamma_1(\Lambda_j^0,t^0)\le\Gamma_2(\Lambda_j^0,t^0)$ at $\partial\Omega$. Then $\Gamma_1(\Lambda,t)\le\Gamma_2(\Lambda,t)$ in Ω.*

By theorem 2, the initial-boundary value $\Gamma(\Lambda_j^0,t^0)$ can be chosen to ensure the a priori bound for solutions of (3.17), theorem 3 also ensures that $\Gamma_3=\Gamma_1-\Gamma_2\le 0$. Hence the existence and uniqueness of classical solutions for (3.16) are proved.

The L^2 theory of equation (3.16) can be stated in the following [97],

Theorem 4: *For $\Gamma\in L^2(\Omega)$, the initial-boundary value problem of (3.16) admits at most one weak solution.*

Proof:

(Uniqueness). Let Γ_1 and Γ_2 be weak solutions of initial-boundary value problem (3.16), if $\Gamma_3=\Gamma_1-\Gamma_2\in W_2^{1,1}(\Omega)$ and fulfill,

$$\iint_{\Omega\times T}\frac{\partial\Gamma_3}{\partial t}\vartheta+\Lambda_j^2\vartheta\vec{\nabla}\Gamma_3-\upsilon\Lambda_j^2\vec{\nabla}\Gamma_3\cdot\vec{\nabla}\vartheta d\Omega dt=0 \tag{3.20a}$$

Choosing $\vartheta=\Gamma_3$ and the maximum principle reveals

$$\iint_{\Omega\times T}\frac{\partial\Gamma_3}{\partial t}\Gamma_3+\Lambda_j^2\Gamma_3\vec{\nabla}\Gamma_3 d\Omega dt=\iint_{\Omega\times T}\upsilon\Lambda_j^2\left|\vec{\nabla}\Gamma_3\right|^2 d\Omega dt\le 0 \tag{3.20b}$$

Poincare inequality is then implemented to obtain

$$\iint_{\Omega\times T}\upsilon\Lambda_j^2\Gamma_3^2 d\Omega dt\le 0 \tag{3.20c}$$

Therefore, $\Gamma_3=0$ and $\Gamma_1=\Gamma_2$ in Ω which ensure the uniqueness of weak solutions.

(Existence). Considering,

$$\iint_{\Omega\times T}\left(\frac{\partial\Gamma}{\partial t}\Gamma+\Lambda_j^2\Gamma\vec{\nabla}\Gamma\right)e^{-\alpha s}d\Omega dt=\iint_{\Omega\times T}\left(\upsilon\Lambda_j^2\left|\vec{\nabla}\Gamma\right|^2\right)e^{-\alpha s}d\Omega dt,\quad s\in\Omega\times T \tag{3.20d}$$

The right hand side is equal to,

$$\iint_{\Omega\times T}\left(\upsilon\Lambda_j^2\left|\vec{\nabla}\Gamma\right|^2\right)e^{-\alpha s}d\Omega dt=\left(\int_{\Omega}\upsilon\Lambda_j^2\left|\vec{\nabla}\Gamma\right|^2 d\Omega\right)\left(\int_T e^{-\alpha s}dt\right)+\alpha\iint_{\Omega\times T}\left(\upsilon\Lambda_j^2\left|\vec{\nabla}\Gamma\right|^2\right)e^{-\alpha s}d\Omega dt \tag{3.20e}$$

and (3.20d) will change as follows,

$$\iint_{\Omega\times T}\left(\frac{\partial\Gamma}{\partial t}\Gamma+\Lambda_j^2\Gamma\vec{\nabla}\Gamma\right)e^{-\alpha s}d\Omega dt\geq\alpha\iint_{\Omega\times T}\left(\upsilon\Lambda_j^2\left|\vec{\nabla}\Gamma\right|^2\right)e^{-\alpha s}d\Omega dt \tag{3.20f}$$

Poincare inequality in the form,

$$\iint_{\Omega\times T}\left(\upsilon\Lambda_j^2\Gamma^2\right)e^{-\alpha s}d\Omega dt\leq M\iint_{\Omega\times T}\left(\upsilon\Lambda_j^2\left|\vec{\nabla}\Gamma\right|^2\right)e^{-\alpha s}d\Omega dt$$

will take us to,

$$\alpha\iint_{\Omega\times T}\left(\upsilon\Lambda_j^2\left|\vec{\nabla}\Gamma\right|^2\right)e^{-\alpha s}d\Omega dt+\frac{1}{M}\iint_{\Omega\times T}\left(\upsilon\Lambda_j^2\Gamma^2\right)e^{-\alpha s}d\Omega dt\leq\iint_{\Omega\times T}\left(\frac{\partial\Gamma}{\partial t}\Gamma+\Lambda_j^2\Gamma\vec{\nabla}\Gamma-\upsilon\Lambda_j^2\left|\vec{\nabla}\Gamma\right|^2\right)e^{-\alpha s}d\Omega dt \tag{3.20g}$$

Therefore, there exist,

$$\|\Gamma\|_{W_2^{1,1}(\Omega)^3}\leq\iint_{\Omega\times T}\left(\frac{\partial\Gamma}{\partial t}\Gamma+\Lambda_j^2\Gamma\vec{\nabla}\Gamma-\upsilon\Lambda_j^2\left|\vec{\nabla}\Gamma\right|^2\right)e^{-\alpha s}d\Omega dt \tag{3.20h}$$

as weak solutions to the initial-boundary value problem of (3.16).

The existence and uniqueness of solutions with intermediate regularity is based on the L^p theory as follows,

Theorem 5: *For* $\Gamma\in L^p(\Omega)$*, the initial-boundary value problem of (3.16) admits a unique strong solution* $\Gamma\in W_p^{2,1}(\Omega)\cap W_p^{1,1}(\Omega)$.

Proof: Multiplying both sides of equation (3.16) by $|\Gamma|^{p-2}\Gamma$ and integrating over Ω and T as,

$$\iint_{\Omega\times T}\frac{\partial\Gamma}{\partial t}|\Gamma|^{p-2}\Gamma+\Lambda_j^2\Gamma|\Gamma|^{p-2}\vec{\nabla}\Gamma d\Omega dt=\iint_{\Omega\times T}\upsilon\Lambda_j^2|\Gamma|^{p-2}\Gamma\vec{\nabla}^2\Gamma d\Omega dt \tag{3.21a}$$

Integrating by parts over spatial coordinate to yield,

$$\frac{1}{p}\iint_{\Omega\times T}\frac{\partial|\Gamma|^p}{\partial t}d\Omega dt+\frac{1}{p}\iint_{\Omega\times T}\Lambda_j^2\vec{\nabla}|\Gamma|^p d\Omega dt=\frac{4(p-1)}{p^2}\iint_{\Omega\times T}\upsilon\Lambda_j^2\left|\vec{\nabla}\left(|\Gamma|^{p/2-1}\Gamma\right)\right|^2 d\Omega dt \tag{3.21b}$$

Multiplying by $e^{-\gamma s}$ as in theorem 3, and by similar procedure the following result is obtained,

$$\|\Gamma\|_{W_p^{2,1}(\Omega)^3 \cap W_p^{1,1}(\Omega)^3} \le \frac{1}{p}\iint_{\Omega\times T} \frac{\partial|\Gamma|^p}{\partial t} d\Omega dt + \frac{1}{p}\iint_{\Omega\times T} \Lambda_j^2 \vec{\nabla}|\Gamma|^p \, d\Omega dt - \frac{4(p-1)}{p^2}\iint_{\Omega\times T} \upsilon\Lambda_j^2 \left|\vec{\nabla}\left(|\Gamma|^{p/2-1}\Gamma\right)\right|^2 d\Omega dt$$

(3.21c)

Thus, suppose that $\Gamma_1,\Gamma_2 \in W_p^{2,1}(\Omega)\cap W_p^{1,1}(\Omega)$ are strong solutions, and based on the estimate of the maximum principle and Poincare inequality, the following is obtained,

$$\frac{4(p-1)}{p^2}\iint_{\Omega\times T} \upsilon\Lambda_j^2\left(|\Gamma|^{p/2-1}\Gamma\right) d\Omega dt \le 0 \qquad (3.21d)$$

Setting $\Gamma_3 = \Gamma_1 - \Gamma_2 = 0$ then $\Gamma_1 = \Gamma_2$ and uniqueness is also proved.

Equation (3.18) is easier to be analysed since $q(x,y,z,t)$ is proved to be bounded. Here the existence and uniqueness of the regular solutions of (3.18) will be demonstrated.

Theorem 6: *Let q be bounded, the boundary value problem (3.18) admits a unique classical solution.*

The proof is similar that of theorem 2.

Theorem 7: *For any $q \in L^2(\Omega)$ and bounded, the boundary value problem (3.18) admits at most one solution.*

Proof: Multiplying (3.18) by ϑ, then there exists a unique $\Phi \in H^1(\Omega)$ such that,

$$\int_\Omega \vec{\nabla}\Phi\cdot\vec{\nabla}\vartheta \, d\Omega = \int_\Omega q\vartheta \, d\Omega, \quad \forall\vartheta \in H^1(\Omega) \qquad (3.22)$$

This shows the existence of the weak solutions of boundary value problem of (3.18).

Theorem 8: *For any $q \in L^p(\Omega)$, equation (3.18) admits a unique strong solution $\Phi \in W^{2,p}(\Omega)\cap W^{1,p}(\Omega)$.*

Proof: The proof here is different than that of theorem 5 since it is guaranteed that $q \in L^p(\Omega)$ is bounded. Multiplying (3.18) by $|\Phi|^{p-2}\Phi$ and the relation over Ω is,

$$\int_{\Omega} |\Phi|^{p-2} \Phi \vec{\nabla}^2 \Phi d\Omega = \int_{\Omega} q |\Phi|^p \Phi d\Omega \tag{3.23a}$$

Integrating,

$$\frac{4(p-1)}{p^2} \int_{\Omega} \left| \vec{\nabla} \left(|\Phi|^{p/2-1} \Phi \right) \right|^2 d\Omega = \int_{\Omega} q |\Phi|^p \Phi d\Omega \tag{3.23b}$$

By Poincare inequality, Holder inequality and Young inequality, equation (3.23b) will transform to,

$$\begin{aligned} \frac{4(p-1)}{Mp^2} \int_{\Omega} |\Phi|^p d\Omega &\le \int_{\Omega} q |\Phi|^{p-1} d\Omega \\ &\le \|q\|_{L^p(\Omega)} \|\Phi\|_{L^p(\Omega)}^{p-1} \\ &\le \varepsilon \int_{\Omega} |\Phi|^p d\Omega + \varepsilon^{-1/(p-1)} \int_{\Omega} |q|^p d\Omega \end{aligned} \tag{3.23c}$$

where C is constant in Poincare inequality and ε is constant in Young inequality, and the above result leads to,

$$\|\Phi\|_{W^{2,p}(\Omega)} \le M \|q\|_{L^p(\Omega)} \tag{3.23d}$$

Let $\Phi_1, \Phi_2 \in W^{2,p}(\Omega) \cap W^{1,p}(\Omega)$ and set $\Phi_3 = \Phi_1 - \Phi_2$ then the estimate (3.23d) will ensure the uniqueness of strong solutions. This proves the theorem.

Hence, the initial-boundary value problem of (3.16) and (3.18) proves have generalised unique solution. If $\Gamma(\Lambda_i)$ is held, then $\Gamma(\Lambda_i)$ will admit general classical solution and will have global regularity for weak solution in L^2 and strong solution in L^p with the assumption of regular boundary.

Chapter 4

Contributions on the Exact Solutions to the Three-Dimensional Incompressible Navier-Stokes Equations

In this section, the contribution to the exact solutions and their properties explained. Unlike in chapter 3, the problems of existence, uniqueness and regularised solution are explained more explicitly using the closed-form solution. This approach is advantageous since the behavior of a mathematical system can easily be described by simulations as well as by a qualitative well-posed problem that is given by exact solution.

The Navier-Stokes equations together with the continuity, basically have many classes of solution since they are nonlinear. The most trivial solutions are zero and constants which have already extensively been explained with their consequences in the previous section. In this chapter, more complex solutions are generated based on the decomposition of the potential function Φ, coordinate transformation, time relation and pressure gradient.

The subject is divided into four sections, the first is by utilising a four components coordinate transformation and no decomposition in the potential function. The formulation is applied either for zero, constant and variable pressure gradient. The second section is by using a three components coordinate transformation with decomposition of the potential function into two variables applied to zero and constant pressure gradient. The third is by implementing two components coordinate transformation with functional time. The potential function is decomposed into three variables and the formulation is applied to variable pressure gradient. The vorticity equations are implemented in the fourth section. In this case, the potential function is also decomposed into three variables together with two components coordinate transformation and functional time.

4.1 Analytical Solution with Four Components Coordinate Transformation

The three-dimensional incompressible Navier Stokes equations is expanded in Cartesian form from (3.1a) as,

At x-direction:
$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\upsilon\frac{\partial^2 u}{\partial x^2}+\upsilon\frac{\partial^2 u}{\partial y^2}+\upsilon\frac{\partial^2 u}{\partial z^2} \tag{4.1a}$$

At y-direction:
$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial y}+\upsilon\frac{\partial^2 v}{\partial x^2}+\upsilon\frac{\partial^2 v}{\partial y^2}+\upsilon\frac{\partial^2 v}{\partial z^2} \tag{4.1b}$$

At z-direction:
$$\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial z}+\upsilon\frac{\partial^2 w}{\partial x^2}+\upsilon\frac{\partial^2 w}{\partial y^2}+\upsilon\frac{\partial^2 w}{\partial z^2} \tag{4.1c}$$

The continuity equation is written as,

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \tag{4.1d}$$

The three velocity components are interlinked and coupled together such as the velocity magnitude in vertical sum can be written as, $\left\|\vec{V}\right\|=\left(u^2+v^2+w^2\right)^{1/2}$.

Consider a potential function Φ, so that its derivatives are the velocity components which are expressed in vectorial form as,

$$\vec{V}=\vec{\nabla}\Phi+\vec{\nabla}\times\Phi \tag{4.2a}$$

where $\vec{\nabla}=<\partial/\partial x,\partial/\partial y,\partial/\partial z>$. The spatial coordinates are transformed into a single coordinate through the following transform function,

$$\xi=kx+ly+mz-ct \tag{4.2b}$$

where k, l, m and c are constants. The velocity components in equation (4.2a) can be rewritten including the new coordinate. Then, substituting to the Navier-Stokes equations and adding them all to give,

$$-A_0\frac{\partial^2\Phi}{\partial\xi^2}+B_0\frac{\partial^2\Phi}{\partial\xi^2}\frac{\partial\Phi}{\partial\xi}=-\frac{C_0}{\rho}\frac{\partial P}{\partial\xi}+D_0 v\frac{\partial^3\Phi}{\partial\xi^3} \tag{4.3}$$

where A_i, B_i, C_i and D_i are constants. If the pressure gradient term is dropped, and the equation is integrated once, solution for $\partial\Phi/\partial\xi$ is obtained. By performing integration once more, the expression for Φ is produced with different constant coefficients as [35],

$$A_1\ln\left(1+e^{B_1\xi}\right)+C_1\xi+D_1 \tag{4.4}$$

Thus, by implementing coordinate relation (4.2b) we arrive at explicit analytical solution.

Now equation (4.3) is recalled back for pressure gradient case,

$$-A_0\frac{\partial^2\Phi}{\partial\xi^2}+B_0\frac{\partial^2\Phi}{\partial\xi^2}\frac{\partial\Phi}{\partial\xi}=-\frac{C_0}{\rho}\frac{\partial P}{\partial\xi}+D_0\nu\frac{\partial^3\Phi}{\partial\xi^3}$$

and can be written with consideration of constant and variable pressure gradient. Implementing $Q=\partial\Phi/\partial\xi$ and taking $Q-\frac{A_0}{B_0}=R$ will result in a shorter expression as,

$$R\frac{\partial R}{\partial\xi}=-\frac{C_0}{B_0\rho}\frac{\partial P}{\partial\xi}+\frac{D_0\nu}{B_0}\frac{\partial^2 R}{\partial\xi^2} \tag{4.5a}$$

Integrating once will yield,

$$\frac{\partial R}{\partial\xi}=\frac{B_0}{2D_0\nu}R^2+\frac{C_0}{\rho}\int_\xi\frac{\partial p}{\partial\xi}d\xi \tag{4.5b}$$

Therefore, the problem falls into the class of Riccati equation. By applying $R=-\frac{2g\nu}{b}\frac{S_\xi}{S}$, equation (4.5) will transform to second order linear equation,

$$\frac{\partial^2 S}{\partial\xi^2}=-\left(\frac{2D_0\nu}{B_0}\right)_\xi\frac{\partial S}{\partial\xi}-\left(\frac{C_0}{\rho}\int_\xi\frac{\partial p}{\partial\xi}d\xi\right)S \tag{4.6}$$

The closed-form solution of (4.6) is obtained by decomposing to be a system of differential equation of second and first order which will be extensively discussed in the section 4.3. By transforming back to R and rearranging $Q=R+\frac{A_0}{B_0}$, the solution for Q is produced and so is for potential function. Therefore, an explicit analytical solution is produced using equation (4.2a – b).

4.2 Analytical Solution with Three Components Coordinate Transformation

In this section, three components coordinate transformation with decomposition of the potential function into two variables are applied to zero and constant pressure gradient.

4.2.1 The Role of Potential Function

Consider a potential function Φ, so that the velocity components are the derivatives of the function and can be expressed as $\vec{V} = \vec{\nabla}\Phi + \vec{\nabla} \times \Phi$. Therefore, the velocity components are expressed as follows,

$$u = \frac{\partial \Phi}{\partial x} + \frac{\partial \Phi}{\partial y} - \frac{\partial \Phi}{\partial z}, v = \frac{\partial \Phi}{\partial y} + \frac{\partial \Phi}{\partial z} - \frac{\partial \Phi}{\partial x}, \text{ and } w = \frac{\partial \Phi}{\partial z} + \frac{\partial \Phi}{\partial x} - \frac{\partial \Phi}{\partial y} \tag{4.7a}$$

The spatial coordinates are transformed into a single coordinate through the following transformation,

$$\xi = ly + mz - ct \tag{4.7b}$$

The above transformation is similar to that given by Mohyuddin et. al. [26]. Velocity components in equation (4.7a) can now be rewritten using the new coordinate,

$$u = \frac{\partial \Phi}{\partial x} + (l - m)\frac{\partial \Phi}{\partial \xi}, \ v = (l + m)\frac{\partial \Phi}{\partial \xi} - \frac{\partial \Phi}{\partial x}, \text{ and } w = \frac{\partial \Phi}{\partial x} + (m - l)\frac{\partial \Phi}{\partial \xi} \tag{4.8}$$

The first step in the derivation is to rewrite the continuity equation in the new notation. Using the velocity components in the new coordinate in equation (4.7b), the continuity equation can be expressed in simpler form.

The potential function is assumed to take the following particular form, which will satisfy the continuity and Navier-Stokes equations,

$$\Phi = P(x,\xi)R(\xi) \tag{4.9a}$$

Substituting equation (4.9a) into the continuity equation will give the following expression,

$$RP_{xx} + A_2 RP_{\xi\xi} + B_2 PR_{\xi\xi} + 2D_2 P_\xi R_\xi = 0 \tag{4.9b}$$

where A_2, B_2 and D_2 are some constants due to the transformation coordinate.

Let $P_{xx} = -A_2 P_{\xi\xi}$, then the last two terms will produce the following equation,

$$\frac{R_{\xi\xi}}{R_\xi} = C_2 \frac{P_\xi}{P} = C_3, \text{ or } R_{\xi\xi} = C_3 R_\xi \tag{4.9c}$$

where C_1 and C_2 are constants. Therefore, the relation $\partial P / \partial \xi$ can be taken equal to $C_3 P$, and the general solution for R can be written as [98],

$$R = \varphi(\xi)\int_\xi \frac{1}{\varphi(\xi)^2} \exp\left[\int_\xi C_2 d\xi\right] d\xi \tag{4.9d}$$

where $\varphi(\xi)$ is taken as a particular solution of R.

The second step in the derivation is to apply equation (4.7a) to velocity in the x-direction in the Navier-Stokes equations,

$$-\varsigma\frac{\partial^2\Phi}{\partial\xi\partial x}-\varsigma(l-m)\frac{\partial^2\Phi}{\partial\xi^2}+\left(\frac{\partial\Phi}{\partial x}+(l-m)\frac{\partial\Phi}{\partial\xi}\right)\left(\frac{\partial^2\Phi}{\partial x^2}+(l-m)\frac{\partial^2\Phi}{\partial x\partial\xi}\right)+l\left((l+m)\frac{\partial\Phi}{\partial\xi}-\frac{\partial\Phi}{\partial x}\right)\left(\frac{\partial^2\Phi}{\partial x\partial\xi}+(l-m)\frac{\partial^2\Phi}{\partial\xi^2}\right)$$
$$+m\left(\frac{\partial\Phi}{\partial x}+(m-l)\frac{\partial\Phi}{\partial\xi}\right)\left(\frac{\partial^2\Phi}{\partial x\partial\xi}+(l-m)\frac{\partial^2\Phi}{\partial\xi^2}\right)=$$
$$\upsilon\left(\frac{\partial^3\Phi}{\partial x^3}+(l-m)\frac{\partial^3\Phi}{\partial x^2\partial\xi}\right)+\upsilon l^2\left(\frac{\partial^3\Phi}{\partial x\partial\xi^2}+(l-m)\frac{\partial^3\Phi}{\partial\xi^3}\right)+\upsilon m^3\left(\frac{\partial^2\Phi}{\partial x\partial\xi^2}+(l-m)\frac{\partial^3\Phi}{\partial\xi^3}\right)$$

(4.10a)

It is noted that equation (4.10a) is performed by dropping the pressure gradient; however, the case of constant pressure gradient will produce similar solutions to that of a zero pressure gradient by employing the same methods. The potential function (4.9a) is substituted in the above equation, and the equation can be rewritten as follows,

$$a_0P_{xxx}+b_0P_{xx}P_x+c_0P_{xx}P+d_0P_{xx}+e_0P_x^2+f_0P_xP+g_0P_x+h_0P^2+i_0P=0 \qquad (4.10b)$$

The next step is to repeat the procedure applied to the x-velocity equation, but this time to the velocity in the y direction will yield the following,

$$a_1P_{xxx}+b_1P_{xx}P_x+c_1P_{xx}P+d_1P_{xx}+e_1P_x^2+f_1P_xP+g_1P_x+h_1P^2+i_1P=0 \qquad (4.11)$$

The same procedure is applied to the z velocity equation, giving,

$$a_2P_{xxx}+b_2P_{xx}P_x+c_2P_{xx}P+d_2P_{xx}+e_2P_x^2+f_2P_xP+g_2P_x+h_2P^2+i_2P=0 \qquad (4.12)$$

Thus equations (4.10b), (4.11) and (4.12) can be combined into a single equation,

$$a_3P_{xxx}+b_3P_{xx}P_x+c_3P_{xx}P+d_3P_{xx}+e_3P_x^2+f_3P_xP+g_3P_x+h_3P^2+i_3P=0 \qquad (4.13)$$

It is noticed that $a_i, b_i, c_i, d_i, e_i, f_i, g_i, h_i$ and i_i (where the subscript i is the constant index) are constants with respect to the x axis, but several are ξ dependent as they are solution of continuity equation. Therefore the derivation above can be stated by the following lemma.

Lemma 3: *Let Φ be a differentiable potential function that is defined as a product of $P(x,\xi)$ and $R(\xi)$, and that relates the velocity vector as $\vec{V}=\vec{\nabla}\Phi+\vec{\nabla}\times\Phi$ over x and ξ, where ξ is transformed coordinate defined in (4.7b). The potential function satisfies* *continuity equation with the condition that* $P_{xx} = -A_2 P_{\xi\xi}$, *where* A_2 *is constant, and reduces the Navier-Stokes equations in the following form*

$$a_3 P_{xxx} + b_3 P_{xx} P_x + c_3 P_{xx} P + d_3 P_{xx} + e_3 P_x^2 + f_3 P_x P + g_3 P_x + h_3 P^2 + i_3 P = 0$$

where $a_3, b_3, c_3, d_3, e_3, f_3,$ *and* g_3 *are constants with respect to the* x *axis.*

4.2.2. Solutions for P

In this section, the solution of equations (4.13) is investigated starting with a particular solution and then extending to more general solutions. It will be shown that the determination of general solutions is related to the obtained particular solutions. It is known that a particular class of the solution of nonlinear differential equations can be obtained by several procedures [99 – 101], so two examples of particular analytical solutions of equation (4.13) will be obtained by different procedures. Integrating equation (4.13) once yields,

$$a_4 P_{xx} P_x + b_4 P_x^3 + c_4 P_x^2 P + d_4 P_x^2 + e_4 P_x^2 P + f_4 P_x P^2 + g_4 P_x P + h_4 P^3 + i_4 P^2 + j_4 = 0 \quad (4.14a)$$

Introducing $Q = P_x$, the equation above will then transform to,

$$a_4 Q^2 \frac{\partial Q}{\partial P} + b_4 Q^3 + c_4 Q^2 P + d_4 Q^2 + e_4 Q^2 P + f_4 Q P^2 + g_4 Q P + h_4 P^3 + i_4 P^2 + j_4 = 0 \quad (4.14b)$$

Then, it is not difficult to verify that Q gives a trivial solution,

$$Q = P_x = a_5 P + b_5 \quad (4.14c)$$

This will produce a solution for P as follows,

$$P = a_6 e^{b_6 x} + \text{constant} \quad (4.14d)$$

For the other procedure, the function $Q = P_x$ may be directly employed in equation (4.13) to give,

$$a_3 Q^2 \frac{\partial^2 Q}{\partial P^2} + b_3 Q^2 \frac{\partial Q}{\partial P} + c_3 \frac{\partial Q}{\partial P} QP + d_3 Q \frac{\partial Q}{\partial P} + e_3 Q^2 + f_3 QP + g_3 Q + h_3 P^2 + i_3 P = 0 \quad (4.15a)$$

Differentiating equation (4.15a) twice with respect to P will result in the following equation,

$$a_3 \left(\frac{\partial Q}{\partial P}\right)^2 \frac{\partial^2 Q}{\partial P^2} + b_3 \left(\frac{\partial Q}{\partial P}\right)^3 + c_3 \left(\frac{\partial Q}{\partial P}\right)^2 + d_3 \frac{\partial Q}{\partial P} \frac{\partial^2 Q}{\partial P^2} + e_3 \left(\frac{\partial Q}{\partial P}\right)^2 + f_3 \frac{\partial Q}{\partial P} + g_3 \frac{\partial^2 Q}{\partial P^2} + h_3 = 0 \quad (4.15b)$$

Note that the differentiation procedure is valid based on the relation of integral and differential equations [102]. Grouping and integrating twice with respect to P will lead to the following expression,

$$P_x = a_7 e^{b_7 P} + c_7 e^{-d_7 P} + e_7 \tag{4.15c}$$

Setting $P = \ln G$, equation (4.15c) transforms to,

$$\frac{G_x}{G} = a_8 G + \frac{b_8}{G} + e_7 \tag{4.15d}$$

Therefore, the solution of P can be obtained easily,

$$P = a_9 \ln\left(\frac{b_9 - c_9 e^x}{e^x - 1}\right) + \text{constant} \tag{4.15e}$$

This is similar to the work of Nugroho et al. [103]. However, equation (4.15b) can be performed by setting $Q_x = N$ and $Q_{xx} = N\,\partial N/\partial Q$, which will give a similar result as equation (4.15e). Therefore, the solution procedures produce the following statement,

Corollary 2: $P = a_6 e^{b_6 x}$ + *constant and* $P = a_9 \ln\left(\frac{b_9 - c_9 e^x}{e^x - 1}\right)$ + *constant are examples of the exact solution of equation (4.13).*

Following the method used in the potential function, the above solutions (4.14d) and (4.15e) will be considered as particular solutions of equation (4.13). Letting U be the particular solution of (4.13) and W be the other solution will generate a more general solution for (4.13) in the following form,

$$P = U + W \tag{4.16}$$

Note that the situation is almost hopeless if the general solution is taken as a product of two respective particular solutions i.e. $P = UW$. Therefore, based on (4.16), equation (4.13) is decomposed by substitution into,

$$\begin{aligned} &a_3 U_{xxx} + a_3 W_{xxx} + b_3 U_{xx} U_x + b_3 U_{xx} W_x + b_3 U_x W_{xx} + b_3 W_{xx} W_x + c_3 U_{xx} U + c_3 U_{xx} W + c_3 U W_{xx} + c_3 W_{xx} W + d_3 U_{xx} \\ &+ d_3 W_{xx} + e_3 U_x^2 + 2 e_3 U_x W_x + e_3 W_x^2 + f_3 U_x U + f_3 U_x W + f_3 U W_x + f_3 W_x W + g_3 U_x + g_3 W_x + h_3 U^2 + 2 h_3 U W \\ &+ h_3 W^2 + i_3 U + i_3 W = 0 \end{aligned} \tag{4.17}$$

Some of the terms above will vanish automatically since they satisfy equation (4.13). Then, the only terms left are,

$$b_3 U_{xx} W_x + b_3 U_x W_{xx} + c_3 U_{xx} W + c_3 U W_{xx} + 2 e_3 U_x W_x + f_3 U_x W + f_3 U W_x + 2 h_3 U W = 0 \tag{4.18}$$

The solutions can be found by linear operator analysis to be $W_{xx} + r_1(x) W_x + r_2(x) W = 0$ since the function U is known. Therefore, equation (4.18) has a general solution as follows [98],

$$W=\eta(x)\int_x \frac{1}{\eta(x)^2}\exp\left[-\int_x r_1(x)dx\right]dx \tag{4.19}$$

where $\eta(x)$ is a particular solution of (4.19) which is clearly dependent on U.

Therefore, according to the solution of continuity, a full solution in terms of the potential function is,

$$\Phi=\{U(x,\xi)+W(x,\xi)\}\left\{\varphi(\xi)\int_\xi \frac{1}{\varphi(\xi)^2}\exp\left[\int_\xi C_3 d\xi\right]d\xi\right\}+\text{constant} \tag{4.20}$$

By implementing the coordinate relation (4.7b) the explicit analytical solution is obtained. Note that the solution for constant pressure gradient is similar to that for zero pressure gradient because there will be a constant term in (4.10b), (4.11) and (4.12), and can be expressed as the same polynomial in (4.13). It is interesting to note that more general solutions to (4.13) can be found by substituting additional terms which then resemble the following

$$P=U+W+........ \tag{4.21}$$

Therefore, the main theorem of this work can be constructed as follows

Theorem 9: *Take $\vec{V}$ as a velocity vector that satisfies the continuity and the Navier-Stokes equations over x and ξ, where the transformed coordinate ξ is defined as $\xi=ly+mz-\varsigma t$, where l, m and ς are constants. The velocity vector is proposed to be in the form $\vec{V}=\vec{\nabla}\Phi+\vec{\nabla}\times\Phi$, where the potential function Φ is defined as a product of $P(x,\xi)$ and $R(\xi)$. If P satisfies the condition $P_{xx}=-A_2P_{\xi\xi}$, where A_2 is a constant, then there exist $U(x,\xi)$ and $W(x,\xi)$ as particular solutions for equation (4.13) and $\varphi(\xi)$ as a particular solution for equation (4.9c). They form the potential function as*

$$\Phi=\{U(x,\xi)+W(x,\xi)+......\}\left\{\varphi(\xi)\int_\xi \frac{1}{\varphi(\xi)^2}\exp\left[\int_\xi C_3 d\xi\right]d\xi\right\}+\textit{constant}$$

4.2.3 Implementation of the Method

Two examples are shown in this section to illustrate the applicability of the Theorem. By considering equation (4.9d), it is not hard to see that if C_4 is a particular solution for R, then the general solution for R is $C_5\exp(C_3\xi)$.

The first example can be constructed directly by considering one of the particular solutions in the corollary, $a_9 \ln\left(\frac{b_9 - c_9 e^x}{e^x - 1}\right)$ as P. The solution then becomes,

$$\Phi = \left\{ a_9 \ln\left(\frac{b_9 - c_9 e^x}{e^x - 1}\right)\right\}\left\{C_5 \exp(C_3 \xi)\right\} + \text{constant} \tag{4.22}$$

The second example comes from the function $U = a_6 e^{b_6 x}$ in corollary which is a particular solution for P. By rearranging equation (4.18) in more regular form, then the following expressions are obtained,

$$r_1(x) = \frac{b_3 U_{xx} + 2e_3 U_x + f_3 U}{b_3 U_x}, \; r_2(x) = \frac{c_3 U_{xx} + 2f_3 U_x + b_3 U}{b_3 U_x} \tag{4.23}$$

By substituting the particular solution $U = a_6 e^{b_6 x}$, it is clear that r_1 and r_2 are constants. Equation (4.18) then has the solution $W = a_{10} e^{b_{10} x}$ which can also be the solution for P. By induction, the other terms can also be generated. Therefore, the expression for the potential function is as follows,

$$\Phi = \left\{ a_6 e^{b_6 x} + a_{10} e^{b_{10} x} + \ldots\ldots \right\}\left\{C_5 \exp(C_3 \xi)\right\} + \text{constant} \tag{4.24}$$

Thus, by implementing $\vec{V} = \vec{\nabla}\Phi + \vec{\nabla}\times\Phi$ to (4.22) and (4.24), the explicit expression for velocity vectors is produced as the solutions to the continuity and three-dimensional Navier-Stokes equations.

4.3 Analytical Solution with Two Components Coordinate Transformation

In this section, two components coordinate transformation with functional time is implemented. The potential function is decomposed into three variables and the formulation is applied to variable pressure gradient.

4.3.1 Method of the Solution

A transformed coordinate with a nontrivial relation with respect to time is applied,

$$\xi = kz - \varsigma(t) \tag{4.25a}$$

where k is a constant. The coordinate transformation above implements the functional form of time instead of linear relation as in section 4.2. Therefore, velocity components in equation (4.7a) can now be rewritten using the new coordinate,

$$u=\frac{\partial\Phi}{\partial x}+\frac{\partial\Phi}{\partial y}-k\frac{\partial\Phi}{\partial\xi},\; v=\frac{\partial\Phi}{\partial y}+k\frac{\partial\Phi}{\partial\xi}-\frac{\partial\Phi}{\partial x},\text{ and } w=k\frac{\partial\Phi}{\partial\xi}+\frac{\partial\Phi}{\partial x}-\frac{\partial\Phi}{\partial y} \tag{4.25b}$$

The potential function is assumed to take the following particular form, which will satisfy the continuity and Navier-Stokes equations,

$$\Phi=P(x,y,\xi)R(y)S(\xi) \tag{4.26}$$

Therefore, the following theorem is produced from the problem statement above,

Theorem 10: *Given $\vec{V}$ is a velocity vector that satisfies the continuity and the Navier-Stokes equations over x, y and ξ, where the transformed coordinate ξ is defined as $\xi=kz-\varsigma(t)$, where k is a constant. The velocity vector is proposed to be in the form $\vec{V}=\vec{\nabla}\Phi+\vec{\nabla}\times\Phi$, where the potential function Φ is defined as a product of $P(x,y,\xi)$, $R(y)$ and $S(\xi)$. If P satisfies the condition $P_{xx}+P_{yy}=-k^2P_{\xi\xi}$, then the Navier-Stokes equations are reduced to the following equation including general pressure gradient γ_3*

$$a_3P_{xxx}+b_3P_{xx}P_x+c_3P_{xx}P+d_3P_{xx}+e_3P_x^2+f_3P_xP+g_3P_x+h_3P^2+i_3P=j_3\gamma_3(x,y,\xi)$$

The continuity equation is also reduced to

$$R_{yy}+2l_1(y)R_y-C_6R=0 \quad and \quad S_{\xi\xi}+C_7l_2(\xi)S_\xi+C_8S=0$$

where C_6, C_7, and C_8 are constants and $a_3, b_3, c_3, d_3, e_3, f_3, g_3, h_3, i_3$ and j_3 are y and ξ dependents. Let the condition $b_3P_{xx}P_x+c_3P_{xx}P+e_3P_x^2+f_3P_xP+h_3P^2=0$ be fulfilled. Therefore, there exist $\chi(x)$ and $A(x)$ as particular solutions of $P(x,y,\xi)$ such that the potential function can be written as,

$$\Phi=\left\{\chi(x)\int_x\left[\frac{1}{\chi(x)^2}e^{-\int_x a_6dx}\int_x e^{\int_x a_6dx}\chi(x)j_4\gamma_3dx\right]dx+A(x)\int_x\frac{1}{A(x)^2}\exp\left[-\int_x r_1(x)dx\right]dx+\ldots\ldots\right\}$$
$$\left\{\alpha(y)\int_y\frac{1}{\alpha(y)^2}\exp\left[-\int_y 2l_1(y)dy\right]dy\right\}\left\{\beta(\xi)\int_\xi\frac{1}{\beta(\xi)^2}\exp\left[-\int_\xi C_7l_2(\xi)d\xi\right]d\xi\right\}+const$$

with $\alpha(y)$ and $\beta(\xi)$ as particular solutions of $R(y)$ and $S(\xi)$, respectively.

Proof of Theorem 10:

Substituting equation (4.26) into continuity equation will give the following expression,

$$RSP_{xx}+RSP_{yy}+k^2RSP_{\xi\xi}+PSR_{yy}+2P_yR_yS+k^2PRS_{\xi\xi}+2kP_\xi RS_\xi=0 \tag{4.27a}$$

Let the condition $P_{xx}+P_{yy}=-k^2P_{\xi\xi}$ be fullfilled, then dividing the above equation by PRS and rearranging will produce,

$$\frac{R_{yy}}{R}+2\frac{P_yR_y}{PR}=-\left(k^2\frac{S_{\xi\xi}}{S}+2k\frac{P_\xi S_\xi}{PS}\right)=C_6 \tag{4.27b}$$

where C_6 is constant. By taking the relations P_y/P and P_ξ/P equal to $l_1(y)$ and $l_2(\xi)$, respectively, the following can be written,

$$R_{yy}+2l_1(y)R_y-C_6R=0 \text{ and } S_{\xi\xi}+C_7l_2(\xi)S_\xi+C_8S=0 \tag{4.27c}$$

where C_7 and C_8 are constants. The general solution for R and S can be written as [98],

$$R=\alpha(y)\int_y\frac{1}{\alpha(y)^2}\exp\left[-\int_y 2l_1(y)dy\right]dy\text{ ,and } S=\beta(\xi)\int_\xi\frac{1}{\beta(\xi)^2}\exp\left[-\int_\xi C_7l_2(\xi)d\xi\right]d\xi \tag{4.27d}$$

where $\alpha(y)$ and $\beta(\xi)$ are taken as particular solutions of $R(y)$ and $S(\xi)$, respectively.

Lemma 4: *Let Φ be a differentiable potential function that is defined as a product of $P(x,y,\xi)$, $R(y)$ and $S(\xi)$ that relates the velocity vector as $\vec{V}=\vec{\nabla}\Phi+\vec{\nabla}\times\Phi$ over x, y and ξ, where ξ is a transformed coordinate defined in (4.25a). The potential function satisfies continuity equation with the condition that $P_{xx}+P_{yy}=-k^2P_{\xi\xi}$, where k is a constant, and reduces the Navier-Stokes equations in the following form*

$$a_3P_{xxx}+b_3P_{xx}P_x+c_3P_{xx}P+d_3P_{xx}+e_3P_x^2+f_3P_xP+g_3P_x+h_3P^2+i_3P=j_3\gamma_3(x,y,\xi)$$

where a_3, b_3, c_3, d_3, e_3, f_3, g_3, h_3, i_3 and j_3 are constants with respect to the x axis.

Proof of Lemma 4: The derivation is to apply equation (4.25a) and (4.25b) to velocity in the x direction in the Navier-Stokes equations,

$$\begin{aligned}
&-\frac{\partial\varsigma}{\partial t}\frac{\partial^2\Phi}{\partial\xi\partial x}-\frac{\partial\varsigma}{\partial t}\frac{\partial^2\Phi}{\partial\xi\partial y}+\frac{\partial\varsigma}{\partial t}k\frac{\partial^2\Phi}{\partial\xi^2}+\left(\frac{\partial\Phi}{\partial x}+\frac{\partial\Phi}{\partial y}-k\frac{\partial\Phi}{\partial\xi}\right)\left(\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial x\partial y}+k\frac{\partial^2\Phi}{\partial x\partial\xi}\right)\\
&+\left(\frac{\partial\Phi}{\partial y}+k\frac{\partial\Phi}{\partial\xi}-\frac{\partial\Phi}{\partial x}\right)\left(\frac{\partial^2\Phi}{\partial x\partial y}+\frac{\partial^2\Phi}{\partial y^2}-k\frac{\partial^2\Phi}{\partial x\partial\xi}\right)+\left(k\frac{\partial\Phi}{\partial\xi}+\frac{\partial\Phi}{\partial x}-\frac{\partial\Phi}{\partial y}\right)\left(k\frac{\partial^2\Phi}{\partial x\partial\xi}+k\frac{\partial^2\Phi}{\partial y\partial\xi}-k^2\frac{\partial^2\Phi}{\partial\xi^2}\right)\\
&=\upsilon\left(\frac{\partial^3\Phi}{\partial x^3}+\frac{\partial^3\Phi}{\partial x^2\partial y}-k\frac{\partial^3\Phi}{\partial x^2\partial\xi}\right)+\upsilon\left(\frac{\partial^3\Phi}{\partial x\partial y^2}+\frac{\partial^3\Phi}{\partial y^3}-k^2\frac{\partial^3\Phi}{\partial x\partial\xi^2}\right)\\
&+\upsilon\left(k^2\frac{\partial^2\Phi}{\partial x\partial\xi^2}+k^2\frac{\partial^3\Phi}{\partial y\partial\xi^2}-k^3\frac{\partial^3\Phi}{\partial\xi^3}\right)+\frac{k}{\rho}\gamma_0(x,y,\xi)
\end{aligned} \tag{4.28a}$$

It is noted that the Navier-Stokes equations are written by performing the pressure gradient as a functional in $\gamma_i(x,y,\xi)$, where the subscript i is the index. The potential function (4.26) is substituted in the above equation, and the equation can be rewritten as follows,

$$a_0P_{xxx}+b_0P_{xx}P_x+c_0P_{xx}P+d_0P_{xx}+e_0P_x^2+f_0P_xP+g_0P_x+h_0P^2+i_0P=j_0\gamma_0(x,y,\xi) \quad (4.28b)$$

The next step is to repeat the procedure applied to the x velocity equation, but this time to the velocity in the y direction, which will yield the following,

$$a_1P_{xxx}+b_1P_{xx}P_x+c_1P_{xx}P+d_1P_{xx}+e_1P_x^2+f_1P_xP+g_1P_x+h_1P^2+i_1P=j_1\gamma_1(x,y,\xi) \quad (4.29)$$

The same procedure is applied to the z velocity equation, giving,

$$a_2P_{xxx}+b_2P_{xx}P_x+c_2P_{xx}P+d_2P_{xx}+e_2P_x^2+f_2P_xP+g_2P_x+h_2P^2+i_2P=j_2\gamma_2(x,y,\xi) \quad (4.30)$$

Thus, equations (4.28b), (4.29) and (4.30) can be combined into a single equation,

$$a_3P_{xxx}+b_3P_{xx}P_x+c_3P_{xx}P+d_3P_{xx}+e_3P_x^2+f_3P_xP+g_3P_x+h_3P^2+i_3P=j_3\gamma_3(x,y,\xi) \quad (4.31)$$

It is noticed that $a_i, b_i, c_i, d_i, e_i, f_i, g_i, h_i, i_i$ and j_i (where the subscript i is the constant index) are some constants with respect to the x axis, but y and ξ dependent as they are solution of continuity equation. This proves lemma 4.

It is known that a particular class of the solution of nonlinear differential equations can be obtained by several procedures [99 – 101]. Let the nonlinear terms in (4.31) satisfy the following condition,

$$b_3P_{xx}P_x+c_3P_{xx}P+e_3P_x^2+f_3P_xP+h_3P^2=0 \quad (4.32a)$$

Introducing $Q=P_x$, the equation above will then transform to,

$$b_3Q^2\frac{\partial Q}{\partial P}+c_3Q\frac{\partial Q}{\partial P}P+f_3QP+h_3P^2=0 \quad (4.32b)$$

Then, it is not hard to verify that Q gives a trivial solution,

$$Q=P_x=a_4P \quad (4.32c)$$

This can be substituted to the remaining terms of (4.31) to yield,

$$a_5P_{xx}+b_5P_x+c_5P=j_3\gamma_3 \quad (4.33a)$$

Therefore, a general solution for (4.33a) is obtained as,

$$P=\chi(x)\int_x\left[\frac{1}{\chi(x)^2}e^{-\int_x a_6dx}\int_x e^{\int_x a_6dx}\chi(x)\,j_4\gamma_3dx\right]dx \quad (4.33b)$$

where $\chi(x)$ is a particular solution of (4.33a). The non homogenous term of (4.33b) will make the explicit result is harder to be evaluated. The first and second derivatives of P are in the following,

$$P_x = \chi_x \int_x \left[\frac{1}{\chi^2} e^{-\int_x a_6 dx} \int_x e^{\int_x a_6 dx} \chi j_4 \gamma_3 dx \right] dx + \frac{1}{\chi} e^{-\int_x a_6 dx} \int_x e^{\int_x a_6 dx} \chi j_4 \gamma_3 dx \quad (4.33c)$$

$$\begin{aligned} P_{xx} = \chi_{xx} \int_x \left[\frac{1}{\chi^2} e^{-\int_x a_6 dx} \int_x e^{\int_x a_6 dx} \chi j_4 \gamma_3 dx \right] dx + \frac{\chi_x}{\chi^2} e^{-\int_x a_6 dx} \int_x e^{\int_x a_6 dx} \chi j_4 \gamma_3 dx \\ - \frac{\chi_x}{\chi^2} e^{-\int_x a_6 dx} \int_x e^{\int_x a_6 dx} \chi j_4 \gamma_3 dx - \frac{a_6}{\chi} e^{-\int_x a_6 dx} \int_x e^{\int_x a_6 dx} \chi j_4 \gamma_3 dx + j_4 \gamma_3 \end{aligned} \quad (4.33d)$$

Note that the second and the third terms of (4.33d) are canceled. Substituting (4.33c) and (4.33d) into (4.33a) to produce,

$$\begin{aligned} (a_5 \chi_{xx} + b_5 \chi_x + c_5 \chi) \int_x \left[\frac{1}{\chi^2} e^{-\int_x a_6 dx} \int_x e^{\int_x a_6 dx} \chi j_4 \gamma_3 dx \right] dx + (b_5 - a_6) \frac{1}{\chi} e^{-\int_x a_6 dx} \int_x e^{\int_x a_6 dx} \chi j_4 \gamma_3 dx \\ + a_5 j_4 \gamma_3 = j_3 \gamma_3 \end{aligned} \quad (4.33e)$$

The first three terms are vanish since χ is a particular solution of (4.33a). Therefore, by taking $a_6 = b_5$ and $j_4 = \frac{j_3}{a_5}$, then it is proved that the general solution (4.33b) satisfy the corresponding second order differential equation (4.33a).

Following the method used in the R and S, the above solution (4.33b) will be considered as a particular solution of equation (4.31). Letting U be the particular solution of (4.31) and W be the other solution will generate a more general solution for (4.31) in the following form,

$$P = U + W \quad (4.34)$$

Therefore, based on (4.34), equation (4.31) is decomposed by substitution into,

$$\begin{aligned} a_3 U_{xxx} + a_3 W_{xxx} + b_3 U_{xx} U_x + b_3 U_{xx} W_x + b_3 U_x W_{xx} + b_3 W_{xx} W_x + c_3 U_{xx} U + c_3 U_{xx} W + c_3 U W_{xx} + c_3 W_{xx} W + d_3 U_{xx} \\ + d_3 W_{xx} + e_3 U_x^2 + 2 e_3 U_x W_x + e_3 W_x^2 + f_3 U_x U + f_3 U_x W + f_3 U W_x + f_3 W_x W + g_3 U_x + g_3 W_x + h_3 U^2 + 2 h_3 U W \\ + h_3 W^2 + i_3 U + i_3 W = j_3 \gamma_3 \end{aligned} \quad (4.35a)$$

Some terms above will vanish automatically since they satisfy (4.31), then, by implementing the same procedure as before, the only terms left are,

$$a_7 U_{xx} W_x + b_7 U_x W_{xx} + c_7 U_{xx} W + d_7 U W_{xx} + e_7 U_x W_x + f_7 U_x W + f_7 U W_x + g_7 U W = 0 \quad (4.35b)$$

The solutions can be found by linear operator analysis to be $W_{xx} + r_1(x) W_x + r_2(x) W = 0$ since the function U is known.

Different from equation (4.33a), the closed form solution of (4.35b) is much more difficult to obtain because it strongly depends on the functions r_1 and r_2. In this work, the method to obtain the solution is proposed and developed by using additional functions which can be calculated based on arbitrary function taken from r_1. Assuming that a particular solution of (4.35b) is written as $A(x)$, substituting the assumed solution into (4.35b) to produce,

$$A_{xx}+r_1(x)A_x+r_2(x)A=0 \tag{4.36a}$$

Multiplying by a function $\lambda(x)$ and take another function $r_3(x)$ from (4.36a) such that the equation above can be written as,

$$\lambda A_{xx}+(\lambda r_1-r_3)A_x+r_3A_x+\lambda r_2A=0 \tag{4.36b}$$

Equation (4.36b) is decomposed to a system of two differential equations,

$$\lambda A_{xx}+(\lambda r_1-r_3)A_x=-D \text{ and } r_3A_x+\lambda r_2A=D \tag{4.36c}$$

Solutions of the above equations are defined as,

$$A=\int_x\left[e^{-\int_x r_1-\frac{r_3}{\lambda}dx}\int_x e^{\int_x r_1-\frac{r_3}{\lambda}dx}\left(\frac{-D}{\lambda}\right)dx\right]dx \text{ and } A=e^{-\int_x\frac{\lambda r_2}{r_3}dx}\int_x e^{\int_x\frac{\lambda r_2}{r_3}dx}\left(\frac{D}{r_3}\right)dx \tag{4.36d}$$

Note that the solutions of A in equation (4.36d) are equal,

$$-\int_x\left[e^{-\int_x r_1-\frac{r_3}{\lambda}dx}\int_x e^{\int_x r_1-\frac{r_3}{\lambda}dx}\left(\frac{D}{\lambda}\right)dx\right]dx=e^{-\int_x\frac{\lambda r_2}{r_3}dx}\int_x e^{\int_x\frac{\lambda r_2}{r_3}dx}\left(\frac{D}{r_3}\right)dx \tag{4.36e}$$

Differentiating the above equation once yield,

$$-e^{-\int_x r_1-\frac{r_3}{\lambda}dx}\int_x e^{\int_x r_1-\frac{r_3}{\lambda}dx}\left(\frac{D}{\lambda}\right)dx=\frac{D}{r_3}-\frac{\lambda r_2}{r_3}e^{-\int_x\frac{\lambda r_2}{r_3}sx}\int_x e^{\int_x\frac{\lambda r_2}{r_3}dx}\left(\frac{D}{r_3}\right)dx \tag{4.36f}$$

By taking $B=\int_x e^{\int_x r_1-\frac{r_3}{\alpha}dx}\left(\frac{D}{\alpha}\right)dx=\int_x e^{\int_x\frac{\alpha r_2}{r_3}dx}\left(\frac{D}{r_3}\right)dx$, then r_3 and α form the relation below,

$$\frac{r_{3x}}{r_3}+r_1-\frac{r_3}{\lambda}=\kappa(x) \text{ and } \frac{\lambda_x}{\lambda}+\frac{\lambda r_2}{r_3}=\kappa(x) \tag{4.36g}$$

Equation (4.36g) will give the solutions for r_3 and λ as,

$$r_3=-\frac{e^{\int_x\chi-r_1dx}}{\int_x\frac{1}{\lambda}e^{\int_x\chi-r_1dx}dx} \text{ and } \lambda=\frac{e^{\int_x\kappa dx}}{\int_x\frac{r_2}{r_3}e^{\int_x\kappa dx}dx} \tag{4.36h}$$

Let $\alpha = r_3$ and equating the above solutions, the expression of r_3 can be determined as,

$$r_3 = \left(\frac{1+r_2}{r_1}\right) e^{-\int_x \frac{r_1 r_2}{1+r_2} dx} e^{\int_x \kappa dx} \tag{4.36i}$$

where χ is taken as an arbitrary function. Now equation (4.36f) is considered in B form,

$$-r_3 B e^{-\int_x r_1 - \frac{r_3}{\lambda} dx} = r_3 B_x e^{-\int_x \frac{\lambda r_2}{r_3} dx} - \lambda r_2 B e^{-\int_x \frac{\lambda r_2}{r_3} dx} \tag{4.36j}$$

Therefore, D is also defined by solving (4.36j) as,

$$D = -r_3 \exp \int_x \left[e^{-\int_x \left(\frac{\lambda r_2}{r_3} - \frac{r_3}{\lambda} - r_1 \right) dx} \right] dx \tag{4.36k}$$

By considering (4.36d), thus the particular solution of (4.35b) is obtained in closed form. Therefore, equation (4.35b) has a general solution as follows [98],

$$W = A(x) \int_x \frac{1}{A(x)^2} \exp\left[-\int_x r_1(x) dx \right] dx \tag{4.37}$$

It is interesting to note that more general solutions of (4.31) can be extended by substituting additional terms which then resemble the following,

$$P = U + W + \ldots\ldots.. \tag{4.38}$$

Therefore, according to the solution of continuity, a full solution in terms of the potential function is,

$$\Phi = \left\{ \chi(x) \int_x \left[\frac{1}{\chi(x)^2} e^{-\int_x a_6 dx} \int_x e^{\int_x a_6 dx} \beta(x)\, j_3 \gamma_3 dx \right] dx + A(x) \int_x \frac{1}{A(x)^2} \exp\left[-\int_x r_1(x) dx \right] dx + \ldots\ldots. \right\}$$
$$\left\{ \alpha(y) \int_y \frac{1}{\alpha(y)^2} \exp\left[-\int_y 2 l_1(y) dy \right] dy \right\} \left\{ \beta(\xi) \int_\xi \frac{1}{\beta(\xi)^2} \exp\left[-\int_\xi C_7 l_2(\xi) d\xi \right] d\xi \right\} + const \tag{4.39}$$

This completes the proof of theorem 10.

Therefore, by using the relations (4.25a) and (4.25b), the exact solution for the three-dimensional incompressible Navier-Stokes equations is obtained.

4.3.2 Uniqueness and Regularised Solution

Note that equations (4.27c), (4.33a) and (4.35b) are second order linear differential equations, thus, by applying boundary and initial conditions, the uniqueness of the solutions in L^2 and L^p can be ensured [97].

Corollary 3: *The boundary value problem of equations (4.27), (4.33a) and (4.35b) satisfying the continuity and the incompressible three-dimensional Navier-Stokes equations is unique.*

The generality of $\varsigma(t)$ can cause the obtained solutions to develop singularity and destroy uniqueness [7]. The boundary value problem (4.27c), (4.33a) and (4.35b) then become unstable and does not depend on Cauchy data anymore. The expression $\varsigma(t)=\frac{1}{T-t}$ can surely make the solution blows up at $t=T$, where T is a constant that depends on the initial condition. The derivative $\partial\varsigma/\partial t$ enters as a coefficient with respect to x in (4.28a). If the particular solution of (4.31) is described by exponential, it depicts $P\to\infty$ as $t\to T$. Hence, it needs some regularisation procedure.

The regularisation procedure that is proposed here depends on finding an approximation expression for $\vec{V}$. If $\vec{V}$ in a blow up solution of the Navier-Stokes equations as $t\to T$, then the modified regularised solution can be written as,

$$\vec{V}^*=\frac{\vec{V}}{\vec{V}+A_3} \tag{4.40a}$$

where A_3 is a very small number. Integrate (4.40a) with respect to $\vec{V}$, the expression for the modified solution is written as follows,

$$\vec{V}-A_3\ln\left(\vec{V}+A_3\right)-\int_{\vec{V}}\frac{M}{N}\left(1-\frac{A_3}{\vec{V}+A_3}\right)d\vec{V}=\int_{\vec{V}^*}\frac{A_3\vec{V}^*}{\left(\vec{V}^*-[1-B_3]\right)^2}d\vec{V}^* \tag{4.40b}$$

where B_3 is an arbitrary small number, $M=\frac{2B_3\vec{V}}{\vec{V}+A_3}+B_3^2-2B_3$ and $N=\left(\frac{\vec{V}}{\vec{V}+A_3}-[1-B_3]\right)^2$.

The modified equation (4.40b) is finite as $t\to T$ and the right hand side converges to V with the controllable error, $A_3\ln\left(\vec{V}+A_3\right)+\int_{\vec{V}}\frac{M}{N}\left(1-\frac{A_3}{\vec{V}+A_3}\right)d\vec{V}$.

4.4 Analytical Solution to the Vorticity Equations

It is reasonable to investigate the Navier-Stokes equations in terms of vorticity equations [29,41]. The second term of (4.2a) will contribute to the irrotationality of the flows and may be related to the vorticity in the system or generated at the boundary. Vorticity is a flow parameter which doesn't propagate instantly, this is a main reason of seeing vorticity as a fundamental quantity of fluid flows.

4.4.1 Exact Solutions

The velocity function in (4.2a) that satisfies the Navier-Stokes equations and the velocity components in (4.25b) are recalled,

$$\vec{V} = \vec{\nabla}\Phi + \vec{\nabla}\times\Phi$$

$$u = \frac{\partial\Phi}{\partial x} + \frac{\partial\Phi}{\partial y} - k\frac{\partial\Phi}{\partial\xi},\ v = \frac{\partial\Phi}{\partial y} + k\frac{\partial\Phi}{\partial\xi} - \frac{\partial\Phi}{\partial x},\ \text{and}\ w = k\frac{\partial\Phi}{\partial\xi} + \frac{\partial\Phi}{\partial x} - \frac{\partial\Phi}{\partial y}$$

The coordinate transformation in (4.25a) is also recalled,

$$\xi = kz - \varsigma(t)$$

Taking curl operation to the Navier Stokes equations, the following vorticity equations are obtained,

$$\frac{\partial\vec{\omega}}{\partial t} + \vec{V}\cdot\vec{\nabla}\vec{\omega} = \vec{\omega}\cdot\vec{\nabla}\vec{V} + \upsilon\vec{\nabla}^2\vec{\omega} \tag{4.41a}$$

with $\vec{\omega} = \vec{\nabla}\times\vec{V}$ and $\vec{\omega} = <\omega_x, \omega_y, \omega_z>$. The vorticity components are distributed in three dimensions, $\omega_x = \omega_x(x,y,z,t)$, $\omega_y = \omega_y(x,y,z,t)$ and $\omega_z = \omega_z(x,y,z,t)$. Note that the pressure term in the Navier-Stokes equations is vanished by curl procedure. It is not hard to see that vorticity components will satisfy,

$$\omega_x = \frac{\partial^2\Phi}{\partial x\partial y} - \frac{\partial^2\Phi}{\partial y^2} - k^2\frac{\partial^2\Phi}{\partial\xi^2} + k\frac{\partial^2\Phi}{\partial x\partial\xi},\ \omega_y = k\frac{\partial^2\Phi}{\partial y\partial\xi} - k^2\frac{\partial^2\Phi}{\partial\xi^2} - \frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial x\partial y},\text{and}$$

$$\omega_z = k\frac{\partial^2\Phi}{\partial x\partial\xi} - \frac{\partial^2\Phi}{\partial x^2} - \frac{\partial^2\Phi}{\partial y^2} + k\frac{\partial^2\Phi}{\partial y\partial\xi} \tag{4.41b}$$

The potential function is assumed to take the following particular form, which will satisfy the continuity and vorticity equations,

$$\Phi = P(x,y,\xi)R(y)S(\xi) \tag{4.42}$$

Therefore, the following theorem is produced from the problem statement above,

Theorem 11: *Given $\vec{V}$ is a velocity vector that satisfies the continuity and the vorticity equations over x, y and ξ, where the transformed coordinate ξ is defined as $\xi = kz - \varsigma(t)$, where k is a constant. The velocity vector is proposed to be in the form $\vec{V} = \vec{\nabla}\Phi + \vec{\nabla}\times\Phi$, where the potential function Φ is defined as a product of $P(x,y,\xi)$, $R(y)$ and $S(\xi)$. Then there exist $U(x,y,\xi)$ and $W(x,y,\xi)$ as particular solutions for the reduced vorticity equation*

$$a_3P_{xxxx} + b_3P_{xxx}P_x + c_3P_{xxx}P + d_3P_{xxx} + e_3P_{xx}P_x + f_3P_{xx}P + g_3P_{xx} + h_3P_x^2 + i_3P_xP + j_3P_x = 0$$

and $R = e^{-\int_y\left[\left(\int_y l_4(y)dy\right)\right]dy}\int_y C_9 e^{\int_y\left[\left(\int_y l_4(y)dy\right)\right]dy}dy$ *and* $S = e^{-\int_\xi\left[\left(\int_\xi l_5(\xi)d\xi\right)\right]d\xi}\int_\xi C_{10} e^{\int_\xi\left[\left(\int_\xi l_5(\xi)d\xi\right)\right]d\xi}d\xi$ *as general solutions for the reduced continuity equation*

$$R_y + \left(\int_y l_4(y)\,dy\right)R = C_9 \quad \text{and} \quad S_\xi + \left(\int_\xi l_5(\xi)\,d\xi\right)S = C_{10}$$

They form a potential function as

$$\Phi = \left\{U(x,y,\xi) + W(x,y,\xi) + ...\right\}\left\{e^{-\int_y\left[\left(\int_y l_4(y)dy\right)\right]dy}\int_y C_9 e^{\int_y\left[\left(\int_y l_4(y)dy\right)\right]dy}dy\right\}$$
$$\left\{e^{-\int_\xi\left[\left(\int_\xi l_5(\xi)d\xi\right)\right]d\xi}\int_\xi C_{10} e^{\int_\xi\left[\left(\int_\xi l_5(\xi)d\xi\right)\right]d\xi}d\xi\right\}$$

where C_9 and C_{10} are constants and $a_3, b_3, c_3, d_3, e_3, f_3, g_3, h_3, i_3$ and j_3 are constants with respect to x axis. The potential function then generates the velocity vector such that there exists a static pressure p which fulfills the following Navier-Stokes equations

$$\frac{\partial\vec{V}}{\partial t} + \vec{V}\cdot\overline{\nabla\vec{V}} = -\frac{1}{\rho}\vec{\nabla}p + \upsilon\vec{\nabla}^2\vec{V}$$

where ρ is a constant fluid density. The resulting velocity vector $\vec{V}$ and static pressure p appear as the solutions of the continuity and the three-dimensional incompressible Navier-Stokes equations.

Proof of theorem 11:

Substituting equation (4.42) into continuity equation will give the following expression,

$$RSP_{xx} + RSP_{yy} + k^2RSP_{\xi\xi} + PSR_{yy} + 2P_yR_yS + k^2PRS_{\xi\xi} + 2k^2P_\xi RS_\xi = 0 \tag{4.43}$$

Lemma 5: *Let P,R and S in (4.43) are separable, then R and S are becoming solutions of the first order linear differential equations as,*

$$R = e^{-\int_y\left[\left(\int_y l_4(y)dy\right)\right]dy}\int_y C_9 e^{\int_y\left[\left(\int_y l_5(y)dy\right)\right]dy}dy \text{ and } S = e^{-\int_\xi\left[\left(\int_\xi l_5(\xi)d\xi\right)\right]d\xi}\int_\xi C_{10} e^{\int_\xi\left[\left(\int_\xi l_5(\xi)d\xi\right)\right]d\xi}d\xi$$

where C_9 and C_{10} are constants.

Proof: Dividing equation (4.43) by PRS and rearranging will produce the following equation,

$$\frac{P_{xx}}{P}+\frac{P_{yy}}{P}+k^2\frac{P_{\xi\xi}}{P}=-\frac{R_{yy}}{R}-2\frac{P_yR_y}{PR}=k^2\frac{S_{\xi\xi}}{S}+2k\frac{P_\xi S_\xi}{PS}+l_3(y,\xi)=l_4(y) \tag{4.44a}$$

Taking, the first relation in the right hand side,

$$R_{yy}+2\frac{P_y}{P}R_y+l_4(y)R=0 \tag{4.44b}$$

Let $\left(2\frac{P_y}{P}\right)_y=l_4(y)$ then (4.44b) can be written as,

$$R_{yy}+\left[\left(\int_y l_4(y)dy\right)R\right]_y=0 \tag{4.44c}$$

Integrating the above equation once to yield the first order relation,

$$R_y+\left(\int_y l_4(y)dy\right)R=C_9 \tag{4.44d}$$

The next step is taking the second relation of (4.44a) as,

$$S_{\xi\xi}+2\frac{P_\xi}{P}S_\xi+l_5(\xi)S=0 \tag{4.44e}$$

where $l_5(\xi)$ is taken as $\frac{l_3(y,\xi)-l_4(y)}{k^2}$. By the same procedure, the above equation is reduced into,

$$S_\xi+\left(\int_\xi l_5(\xi)d\xi\right)S=C_{10} \tag{4.44f}$$

Therefore, the general solutions for R and S can be written as [98],

$$R = e^{-\int_y\left[\left(\int_y l_4(y)dy\right)\right]dy}\int_y C_9 e^{\int_y\left[\left(\int_y l_5(y)dy\right)\right]dy}dy,\text{ and } S = e^{-\int_\xi\left[\left(\int_\xi l_5(\xi)d\xi\right)\right]d\xi}\int_\xi C_{10} e^{\int_\xi\left[\left(\int_\xi l_5(\xi)d\xi\right)\right]d\xi}d\xi \tag{4.44g}$$

where C_9 and C_{10} are integration constants. This proves lemma 5.

Thus, R and S can be substituted to the potential function (4.42) to produce the following statement,

Lemma 6: *Let Φ be a differentiable potential function that is defined as a product of $P(x,y,\xi)$, $R(y)$ and $S(\xi)$ that relates the velocity vector as $\vec{V}=\vec{\nabla}\Phi+\vec{\nabla}\times\Phi$ over x, y and ξ, where ξ is transformed coordinate defined in (4.25a). The potential function satisfies continuity equation with the condition and reduces the Vorticity equations in the following form*

$$a_3P_{xxxx}+b_3P_{xxx}P_x+c_3P_{xxx}P+d_3P_{xxx}+e_3P_{xx}P_x+f_3P_{xx}P+g_3P_{xx}+h_3P_x^2+i_3P_xP+j_3P_x=0$$

where a_3, b_3, c_3, d_3, e_3, f_3, g_3, h_3, i_3 and j_3 are constants with respect to the x axis.

Proof: Furthermore, the derivation is to apply equation (4.41b) into vorticity equations in x component,

$$\begin{aligned}
&-\frac{\partial\varsigma}{\partial t}\frac{\partial^3\Phi}{\partial x\partial y\partial\xi}+\frac{\partial\varsigma}{\partial t}\frac{\partial^3\Phi}{\partial y^2\partial\xi}+\frac{\partial\varsigma}{\partial t}k^2\frac{\partial^3\Phi}{\partial\xi^3}-\frac{\partial\varsigma}{\partial t}k\frac{\partial^3\Phi}{\partial x\partial\xi^2}+\left(\frac{\partial\Phi}{\partial x}+\frac{\partial\Phi}{\partial y}-k\frac{\partial\Phi}{\partial\xi}\right)\left(\frac{\partial^3\Phi}{\partial x^2\partial y}-\frac{\partial^3\Phi}{\partial x\partial y^2}-k^2\frac{\partial^3\Phi}{\partial x\partial\xi^2}+k\frac{\partial^3\Phi}{\partial x^2\partial\xi}\right)\\
&+\left(\frac{\partial\Phi}{\partial y}+k\frac{\partial\Phi}{\partial\xi}-\frac{\partial\Phi}{\partial x}\right)\left(\frac{\partial^3\Phi}{\partial x\partial y^2}-\frac{\partial^3\Phi}{\partial y^3}-k^2\frac{\partial^3\Phi}{\partial y\partial\xi^2}+k\frac{\partial^3\Phi}{\partial x\partial y\partial\xi}\right)\\
&+\left(k\frac{\partial\Phi}{\partial\xi}+\frac{\partial\Phi}{\partial x}-\frac{\partial\Phi}{\partial y}\right)\left(k\frac{\partial^3\Phi}{\partial x\partial y\partial\xi}-k\frac{\partial^3\Phi}{\partial y^2\partial\xi}-k^3\frac{\partial^3\Phi}{\partial\xi^3}+k^2\frac{\partial^3\Phi}{\partial x\partial\xi^2}\right)=\\
&\upsilon\left(\frac{\partial^4\Phi}{\partial x^3\partial y}-\frac{\partial^4\Phi}{\partial x^2\partial y^2}-k^2\frac{\partial^4\Phi}{\partial x^2\partial\xi^2}+k\frac{\partial^4\Phi}{\partial x^3\partial\xi}\right)+\upsilon\left(\frac{\partial^4\Phi}{\partial x\partial y^3}-\frac{\partial^4\Phi}{\partial y^4}-k^2\frac{\partial^4\Phi}{\partial y^2\partial\xi^2}+k\frac{\partial^4\Phi}{\partial x\partial y^2\partial\xi}\right)\\
&+\upsilon\left(k^2\frac{\partial^4\Phi}{\partial x\partial y\partial\xi^2}-k^2\frac{\partial^4\Phi}{\partial y^2\partial\xi^2}-k^4\frac{\partial^4\Phi}{\partial\xi^4}+k^3\frac{\partial^4\Phi}{\partial x\partial\xi^3}\right)
\end{aligned}\tag{4.45a}$$

The potential function (4.42) is substituted in the above equation, and the equation can be rewritten as,

$$a_0P_{xxx}+b_0P_{xx}P_x+c_0P_{xx}P+d_0P_{xx}+e_0P_x^2+f_0P_xP+g_0P_x+h_0P^2+i_0P=0\tag{4.45b}$$

The next step now is to repeat the procedure applied to the x component of the vorticity equation, but this time to the vorticity in the y component, will yield the following,

$$a_1P_{xxxx}+b_1P_{xxx}P_x+c_1P_{xxx}P+d_1P_{xxx}+e_1P_{xx}P_x+f_1P_{xx}P+g_1P_{xx}+h_1P_x^2+i_1P_xP+j_1P_x+l_1P^2+m_1P=0\tag{4.46}$$

The same procedure is applied to the z component of the vorticity equation, giving,

$$a_2P_{xxxx}+b_2P_{xxx}P_x+c_2P_{xxx}P+d_2P_{xxx}+e_2P_{xx}P_x+f_2P_{xx}P+g_2P_{xx}+h_2P_x^2+i_2P_xP+j_2P_x+l_2P^2+m_2P=0\tag{4.47}$$

Substituting equations (4.45b) and (4.46) into (4.47) to eliminate P^2 and P as follows,

$$a_3 P_{xxxx} + b_3 P_{xxx} P_x + c_3 P_{xxx} P + d_3 P_{xxx} + e_3 P_{xx} P_x + f_3 P_{xx} P + g_3 P_{xx} + h_3 P_x^2 + i_3 P_x P + j_3 P_x = 0 \tag{4.48}$$

It is noticed that $a_i, b_i, c_i, d_i, e_i, f_i, g_i, h_i, i_i, j_i, l_i$ and m_i (where the subscript i is the constant index) are some constants with respect to the x axis, but several are y and ξ dependent as they are solution of continuity equation. This proves lemma 6.

Letting U be the known particular solution of (4.48) and W be the other solution will generate a more general solution for (4.48) in the following form,

$$P = U + W \tag{4.49}$$

Therefore, based on (4.49), equation (4.48) is decomposed by substitution into,

$$\begin{aligned}
&a_3 U_{xxxx} + a_3 W_{xxxx} + b_3 U_{xxx} U_x + b_3 U_{xxx} W_x + b_3 U_x W_{xxx} + b_3 W_{xxx} W_x + c_3 U_{xxx} U + c_3 U_{xxx} W + c_3 U W_{xxx} + c_3 W_{xxx} W \\
&+ d_3 U_{xxx} + d_3 W_{xxx} + e_3 U_{xx} U_x + e_3 U_{xx} W_x + e_3 U_x W_{xx} + e_3 W_{xx} W_x + f_3 U_{xx} U + f_3 U_{xx} W + f_3 U W_{xx} + f_3 W_{xx} W \\
&+ g_3 U_{xx} + g_3 W_{xx} + h_3 U_x^2 + 2h_3 U_x W_x + h_3 W_x^2 + i_3 U_x U + i_3 U_x W + i_3 U W_x + i_3 W_x W + j_3 U_x + j_3 W_x = 0
\end{aligned} \tag{4.50}$$

Some terms above will vanish automatically since they satisfy (4.48), then, the only terms left are,

$$\begin{aligned}
&a_3 W_{xxxx} + b_3 U_{xxx} W_x + b_3 U_x W_{xxx} + b_3 W_{xxx} W_x + c_3 U_{xxx} W + c_3 U W_{xxx} + c_3 W_{xxx} W + d_3 W_{xxx} + e_3 U_{xx} W_x + e_3 U_x W_{xx} \\
&+ e_3 W_{xx} W_x + f_3 U_{xx} W + f_3 U W_{xx} + f_3 W_{xx} W + g_3 W_{xx} + h_3 U_x^2 + 2h_3 U_x W_x + h_3 W_x^2 + i_3 U_x W + i_3 U W_x + i_3 W_x W \\
&+ j_3 W_x = 0
\end{aligned} \tag{4.51}$$

It is interesting to note that more general solutions to (4.48) can be found by substituting additional terms which then resemble the following,

$$P = U + W + \ldots\ldots\ldots \tag{4.52}$$

Therefore, according to the solution of continuity, a full solution in terms of the potential function is,

$$\begin{aligned}
\Phi = &\left\{ U(x,y,\xi) + W(x,y,\xi) + \ldots.. \right\} \left\{ e^{-\int_y \left[\left(\int_y l_4(y)dy\right)\right]dy} \int_y C_9 e^{\int_y \left[\left(\int_y l_4(y)dy\right)\right]dy} dy \right\} \\
&\left\{ e^{-\int_\xi \left[\left(\int_\xi l_5(\xi)d\xi\right)\right]d\xi} \int_\xi C_{10} e^{\int_\xi \left[\left(\int_\xi l_5(\xi)d\xi\right)\right]d\xi} d\xi \right\}
\end{aligned} \tag{4.53}$$

By implementing the coordinate relation (4.25a) the exact solution is obtained.

Now the resulting velocity vectors can be produced by applying equation (4.2a). Then, substituting the velocity vectors into the Navier-Stokes equations to obtain the pressure. This completes the proof of theorem 11.

However, the pressure relation can also be applied to the modified Navier-Stokes equations

$$\frac{1}{\rho}\vec{\nabla}^2 p = -\vec{\nabla}\vec{V}\cdot\vec{\nabla}\vec{V} \tag{4.54}$$

The other terms are dropped by the continuity equation. It can easily be noticed that by substituting the known expressions of the previous result for velocity, equation (4.54) becomes a linear partial differential equation and the pressure relation is also solved by this procedure.

4.4.2 Implementation of the Theorem

One of the crucial problems in the theory of differential equations is finding and studying classes of important equations that are integrable in closed form and, in particular, possess explicit solutions. It is known that a particular class of the solution of nonlinear differential equations can be obtained by several procedures. Introducing $Q = P_x$, equation (4.48) will then transform to,

$$\begin{aligned}&a_3Q\left(\frac{\partial Q}{\partial P}\right)^3 + a_4Q^2\frac{\partial Q}{\partial P}\frac{\partial^2 Q}{\partial P^2} + a_3Q^3\frac{\partial^3 Q}{\partial P^3} + b_3Q^2\left(\frac{\partial Q}{\partial P}\right)^2 + b_3Q^3\frac{\partial^2 Q}{\partial P^2} + c_3QP\left(\frac{\partial Q}{\partial P}\right)^2 + c_3Q^2P\frac{\partial^2 Q}{\partial P^2}\\&+d_3Q\left(\frac{\partial Q}{\partial P}\right)^2 + d_3Q^2\frac{\partial^2 Q}{\partial P^2} + f_3QP\frac{\partial Q}{\partial P} + g_3Q\frac{\partial Q}{\partial P} + h_3Q^2 + i_3QP + j_3Q = 0\end{aligned} \tag{4.55}$$

Considering the following polynomial Q and substitute into (4.55),

$$Q = P_x = a_5P^2 + b_5P + c_5 \tag{4.56a}$$

If the coefficients in (4.56a) are taken as arbitrary values, then the following system is produced from (4.55),

$$\begin{aligned}&Q = P_x = a_5P^2 + b_5P + c_5\\&a_6P^4 + b_6P^3 + c_6P^2 + d_6P + e_6 = 0\end{aligned} \tag{4.56b}$$

For more general solution, equation (4.49) is substituted into (4.48) and will generate the relation below,

$$\begin{aligned}&a_3W_{xxxx} + b_3W_{xxx}W_x + c_3W_{xxx}W + r_1(x)W_{xxx} + e_3W_{xx}W_x + f_3W_{xx}W + r_2(x)W_{xx} + h_3W_x^2 + i_3W_xW\\&+r_3(x)W_x = 0\end{aligned} \tag{4.57a}$$

where $k_1(x), k_2(x)$ and $k_3(x)$ are clearly dependent on U. By applying the same method, the corresponding equation is then,

$$\begin{aligned}&W_x = n_1(x)W^2 + n_2(x)W + n_3(x)\\&n_4(x)W^4 + n_5(x)W^3 + n_6(x)W^2 + n_7(x)W + n_8(x) = 0\end{aligned} \tag{4.57b}$$

Lemma 7: *Let equation (4.57b) be rearranged into a single polynomial differential equation of fourth order,*

$W_x = n_9(x)W^4 + n_{10}(x)W^3 + n_{11}(x)W^2 + n_{12}(x)W$

The above equation has a solution expressed as,

$$W = k_2(x)H(x) = \left\{ \frac{-n_{10} \pm \left[n_{10}^2 - 4n_9\left(n_{11} - \frac{q}{H^2} \right) \right]^{\frac{1}{2}}}{2n_9} \right\} = e^{-\int_x n_{12}dx} \int_x \left(\frac{q}{H^2} n_{12} e^{\int_x n_{12}dx} \right) dx$$

where $H(x)$ *is an arbitrary function. The term* $q(x)$ *depends on* $n_9(x), n_{10}(x), n_{11}(x)$ *and* $n_{12}(x)$*, which is defined as a solution of second order polynomial differential equation.*

Proof: Now equation (4.57b) is considered, by eliminating $n_3(x)$, then a single equation is produced,

$$W_x = n_9(x)W^4 + n_{10}(x)W^3 + n_{11}(x)W^2 + n_{12}(x)W \tag{4.58a}$$

Let $W = F(x)H(x)$, then (4.58a) will become,

$$HF_x + H_xF = n_9(x)H^4F^4 + n_{10}(x)H^3F^3 + n_{11}(x)H^2F^2 + n_{12}(x)HF \tag{4.58b}$$

Factoring the right side of (4.58b) as,

$$\begin{aligned} HF_x + H_xF &= \{F + h_1\}\left\{n_9H^4F^3 + \left(n_{10}H^3 - n_9H^4h_1\right)F^2 + \left(n_{11}H^2 - n_{10}H^3h_1 + n_9H^4h_1^2\right)F\right\} \\ &+ \left\{[h_1 + h_2]\left[-n_9H^4h_1^2 + \left(n_{10}H^3 + n_9H^4h_2\right)h_1 + \frac{n_{12}H}{h_2}\right] + \left[-n_{11}H^2 - n_{10}H^3h_2 - n_9H^4h_2^2\right]h_1\right\}F \\ &+ \left\{+\left[-\frac{n_{12}H}{h_2}\right]h_1\right\}F \end{aligned} \tag{4.58c}$$

Let the solution of (4.58c) taken as $F = -h_1(x) = h_2(x)$ then the equation above becomes,

$$Hh_{2x} + H_xh_2 = \left\{\left[n_{11}H^2 + n_{10}H^3h_2 + n_9H^4h_2^2\right]h_2 + \left[\frac{n_{12}H}{h_2}\right]h_2\right\}h_2 \tag{4.58d}$$

Take the first term on the right side as $n_{11}H^2 + n_{10}H^3h_2 + n_9H^4h_2^2 = q(x)$, where $q(x)$ is an arbitrary function. Thus, the solution for n_2 can be generated easily as,

$$h_2 = \frac{1}{H}\left\{\frac{-n_{10} \pm \left[n_{10}^2 - 4n_9\left(n_{11} - \frac{q}{H^2}\right)\right]^{\frac{1}{2}}}{2n_9}\right\} \tag{4.58e}$$

Equation (4.58d) is rearranged to be,

$$h_{2x} = \frac{q}{H}h_2^2 + \left(n_{12} - \frac{H_x}{H}\right)h_2 \tag{4.58f}$$

Let $h_2 = h_3(x)s_1(x)$, thus the above equation becomes,

$$s_1 h_{3x} + s_{1x}h_3 = \frac{q}{H}s_1^2 h_3^2 + \left(n_{12} - \frac{H_x}{H}\right)s_1 h_3 \tag{4.58g}$$

The same procedure is applied and (4.58g) is factorised to be,

$$s_1 h_{3x} + s_{1x}h_3 = \left\{\left[h_3 + h_4\right]\left[\frac{q}{H}s_1^2 h_3\right] + \left[\left(n_{12} - \frac{H_x}{H}\right)s_1 - \frac{q}{H}s_1^2 h_4\right]h_3\right\} \tag{4.58h}$$

Take the solution of (4.58h) as $h_3 = -h_4$ and perform $\left(n_{12} - \frac{H_x}{H}\right)s_1 - \frac{q}{H}s_1^2 h_4 = s_2(x)$, where $s_2(x)$ will be determined later. Equation (4.58h) thus becomes,

$$h_{4x} = \left(\frac{s_2 - s_{1x}}{s_1}\right)h_4 \tag{4.58i}$$

It is not hard to see that the expression below,

$$h_4 = \frac{1}{s_1}e^{\int_x \frac{s_2}{s_1}dx} = \left(n_{12} - \frac{H_x}{H}\right)\frac{H}{qs_1} - \frac{Hs_2}{qs_1^2} \tag{4.58j}$$

is solution of (4.58i). Let $\ln\theta = \int_x \frac{s_2}{s_1}dx$, then (4.58j) will transformed to second order polynomial differential. Solving for $s_2(x)$ in (4.58j) will make h_4 is represented by $H(x)$, $q(x)$ and $s_1(x)$ as,

$$h_4 = \frac{1}{s_1 H}e^{-\int_x n_{12}dx}\int_x\left(\frac{q}{H^2}n_{12}e^{\int_x n_{12}dx}\right)dx \tag{4.58k}$$

where $H(x)$ and $s_1(x)$ are arbitrary functions. Thus, h_2 is also defined as,

$$h_2 = \frac{1}{H}e^{-\int_x n_{12}dx}\int_x\left(\frac{q}{H^2}n_{12}e^{\int_x n_{12}dx}\right)dx \tag{4.58l}$$

Substituting back to (4.58e), the expression for $q(x)$ can also be obtained as a solution of second order polynomial differential equation. Therefore, the solution of W is,

$$W = h_2(x)H(x) = \left\{ \frac{-n_{10} \pm \left[n_{10}^2 - 4n_9 \left(n_{11} - \frac{q}{H^2} \right) \right]^{\frac{1}{2}}}{2n_9} \right\} = e^{-\int_x n_{12} dx} \int_x \left(\frac{q}{H^2} n_{12} e^{\int_x n_{12} dx} \right) dx \qquad (4.58\text{m})$$

This proves lemma 7.

Lemma 7 is also applied for (4.56b), and the solution of P is obtained in a functional series. It is interesting to note that higher order polynomial equations can also be produced by the proposed procedure through factoring their polynomials and integrating their terms as the keystones.

It is not hard to see that equation (4.55) also admits the condition, $Q = P_x = a_7 P + b_7$ and has a simple solution as,

$$P = e^{\int_x a_7 dx} \int_x b_7 e^{-\int_x a_7 dx} dx \qquad (4.59)$$

Therefore, with lemma 7, the following statement is produced,

Theorem 12: *(Uniqueness) The initial boundary value problem of (4.1), (4.2a) and (4.25a) has unique point values.*

Proof of theorem 12:

By applying the velocity vector (4.2a) and the reverse transformation (4.25a), combining all parts of the potential function and substituting the initial boundary values in the resulting potential function, the solution constants can be obtained. Note that the resulting velocity vector must be the same for the corresponding potential function from (4.58l) and (4.59) to ensure uniqueness. By substituting arbitrary values t^*, y^* and z^* in the solution, a unique value for x^* is found [104]. The process then can be repeated by induction to find any other unique points. This completes the proof of theorem 12.

Chapter 5

Preliminary Validation Cases of the Analytical Solution

The analytical solution is validated at early stage and the validation cases are presented in this chapter. Following the analysis in the previous chapter, the explicit solutions in the proposed class of the respective potential function can vary. Thus, the applicability of a particular function as a solution might be valid for only certain cases. For example, if $P_x = a_7 e^{b_7 P} + c_7 e^{-d_7 P} + e_7$ in (4.15c) is considered, solution for P will be,

$$P = a_9 \ln\left(\frac{b_9 - c_9 e^x}{e^x - 1}\right) + const \qquad (5.1)$$

which is then substituted into the potential function (4.9a).

If the case of decayed velocity of a nozzle exit is considered, then the velocity in x direction parallel to the flow inlet can be approximated as $u \sim \tanh(x)$. Note that the other terms in y and z axis are considered as constants as a consequence of the boundary conditions if the described solution is to be at the centerline of the system.

The redecomposition of the potential function will also contribute to different forms of explicit solutions. As obvious examples, equation (4.22) and (4.24) can be arranged differently as other solutions from the Navier-Stokes equations by implementing,

$$\Phi = P(\xi) R(x,\xi)$$

Applying the same procedure will lead to different solutions as follows,

$$\Phi = \left\{C_5 \exp(C_3 \xi)\right\}\left\{a_9 \ln\left(\frac{b_9 - c_9 e^x}{e^x - 1}\right)\right\} + \text{constant} \qquad (5.2)$$

and

$$\Phi = \left\{C_5 \exp(C_3 \xi)\right\}\left\{a_6 e^{b_6 x} + a_{10} e^{b_{10} x} + \ldots\ldots\right\} + \text{constant} \qquad (5.3)$$

The first terms in (5.2) and (5.3) can also be replaced by trigonometric functions by reexamining the solutions of continuity equations. In each validation case, different types of solutions are chosen based on the systems considered in the case.

There is also an interesting aspect of the solution concerning the initial value problems. The rich structure of the coordinate transformation of time $\varsigma(t)$ allows the investigation of several conditions. For example, by taking $\varsigma(t)=\alpha t$, the solution will not blow up as the constant α is positive value greater than zero. The other condition $\varsigma(t)=\frac{\alpha}{t-T}$, will also be smooth for $\alpha > 0$ and $T > 0$. For negative value of α, the first case will blow up at $t=\infty$, but in the second case, blow up can be at very short time $t=T$. This, however, is more on the mathematical problems of the solutions and will not be considered in the validations.

5.1 Laminar Flow Cases

The first validation case is the laminar free jet experiment of Symons and Labus [105]. A jet is the flow generated by a continuous source of momentum. The Reynolds number of a jet can be conveniently defined as $\mathrm{Re}=\frac{u_0 D}{\nu}$, where u_0 is jet velocity, D is jet diameter and ν is kinematic viscosity of the fluid. The prescribed data is the normalised downstream velocity. Fig. 5.1 shows a comparison between calculated centerline downstream velocity using the analytical solution and measured velocity. As shown, the analytical solution could reproduce the decay of the measured downstream velocity with longitudinal distance from the nozzle. In the figure, both experimental data and analytical calculations are normalised. It is observed that the comparison for higher velocity (lower figure) is more accurate. It may be due to the characteristic of the solution itself. Analytical solutions are obtained through the simple coordinate transformation. By dimensional analysis, it is clear that contributions of viscous terms are weakened for higher Reynolds number as described below,

$$\frac{\partial U}{\partial \eta}+U.\nabla U=-\nabla P+\frac{1}{\mathrm{Re}}\nabla^2 U \tag{5.4}$$

with $\Omega=Wt$, $\eta=\Omega/L$, $U=u/W$ and $\mathrm{Re}=WL/\nu$.

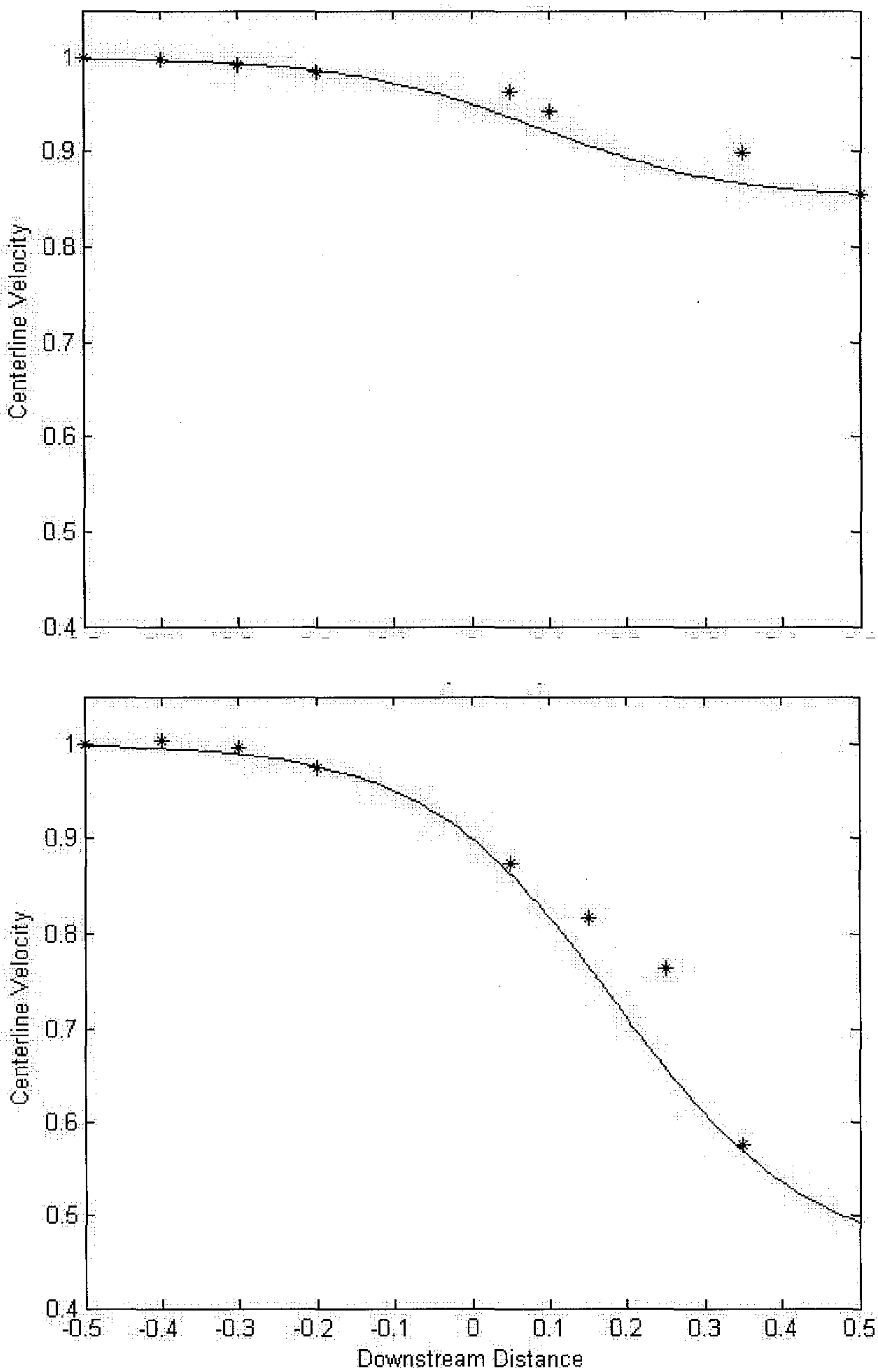


Figure 5.1: Decaying velocity along downstream direction produced by analytical solution (solid line) and the experimental [105] data (points).

More comparison of the velocity profile is shown in Fig. 5.2. The transverse velocity is the one used for comparison here. The calculated values follow the same trend as the measured ones with high accuracy. However, for some points far from centerline there are slight deviations which can be attributed to the vortex formation around the longitudinal axis immediately when the flow jets out of the nozzle exit.

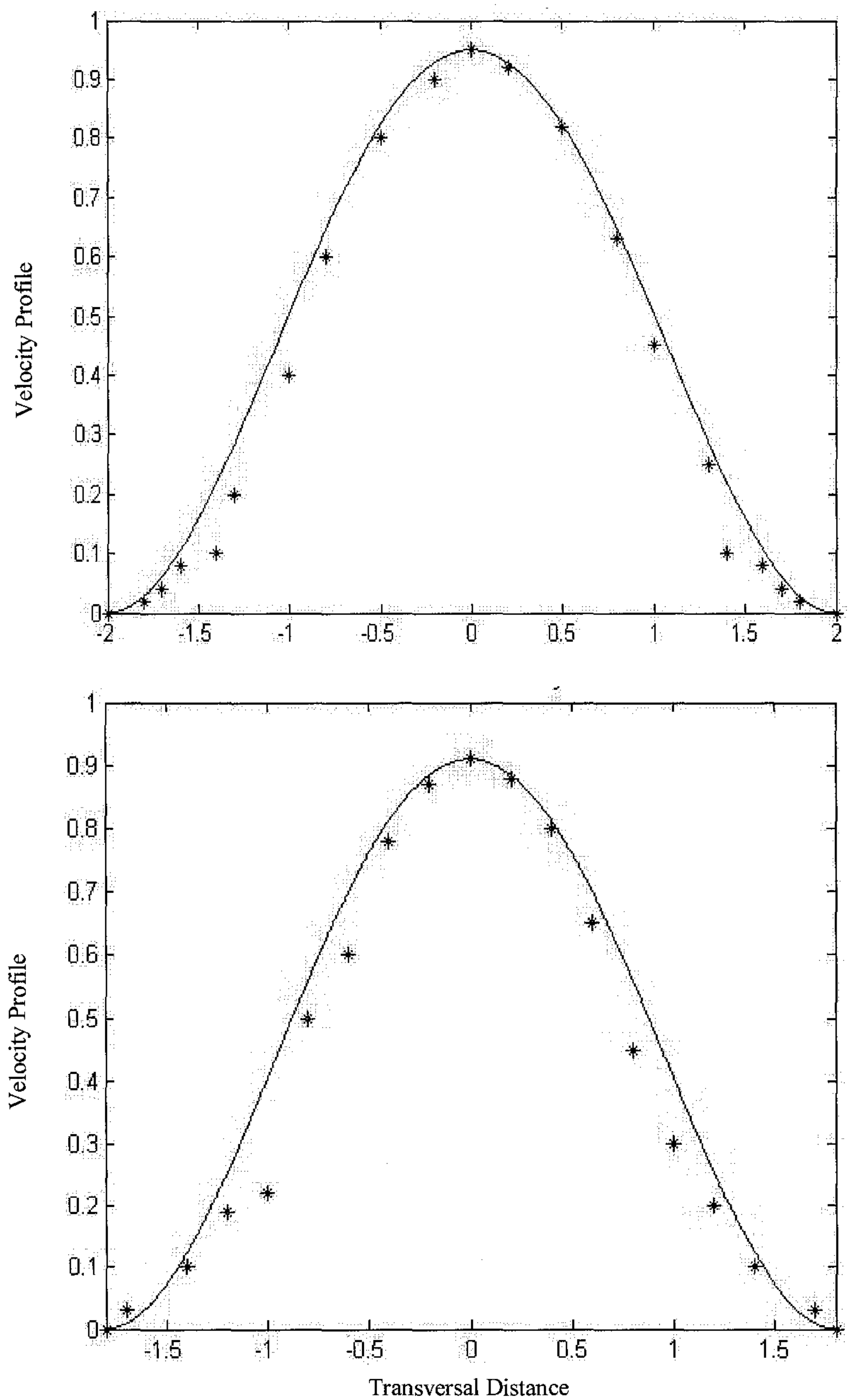
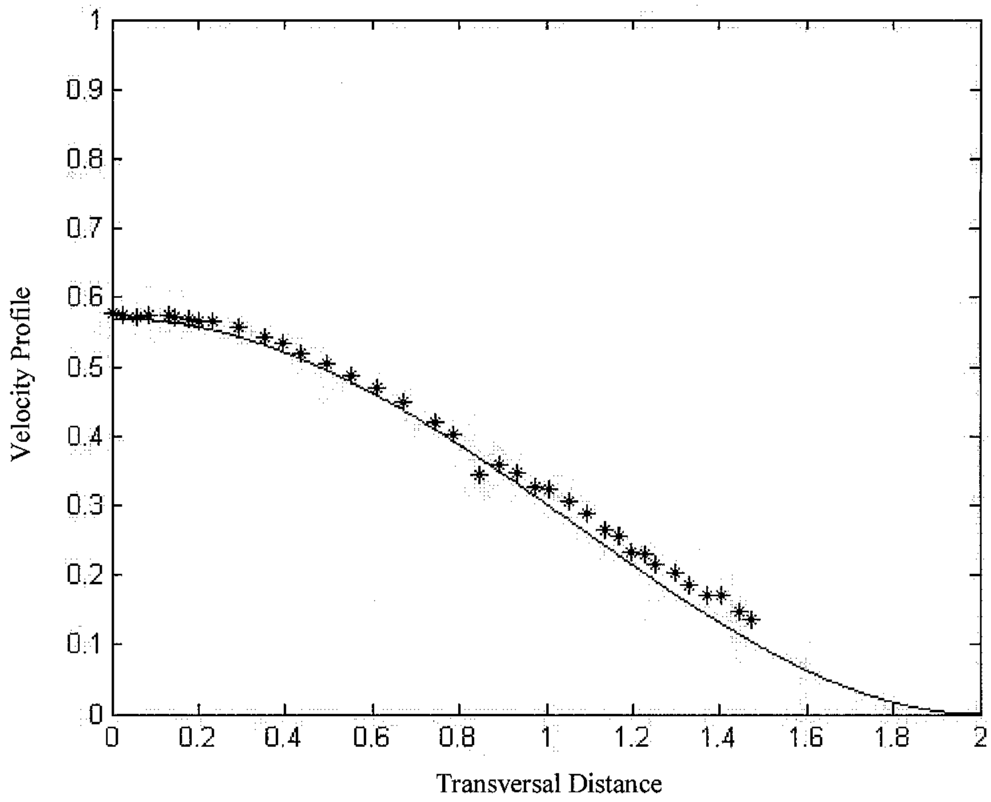


Figure 5.2: Gaussian velocity along transversal distance produced by analytical solution (solid line) and the experimental [105] data (points).

The other similar experiment used for validation is the laminar free jet of Eappen [106]. Fig. 5.3 shows a comparison of the velocity profile calculated using the analytical solution predictions with the measured values. The inlet boundary condition is based on parabolic velocity profile to match the experimental set up. Different from the decay velocity, comparisons for transverse velocity profile show that calculation for higher velocity (lower figure) is less accurate than the other. This might happen due to the vortex formation of the flow as it leaves the nozzle. The vortex formation has a general tendency to produce and accumulate the eddies along the longitudinal direction. The process is happen through entrainment which draws the material outside into the jet.



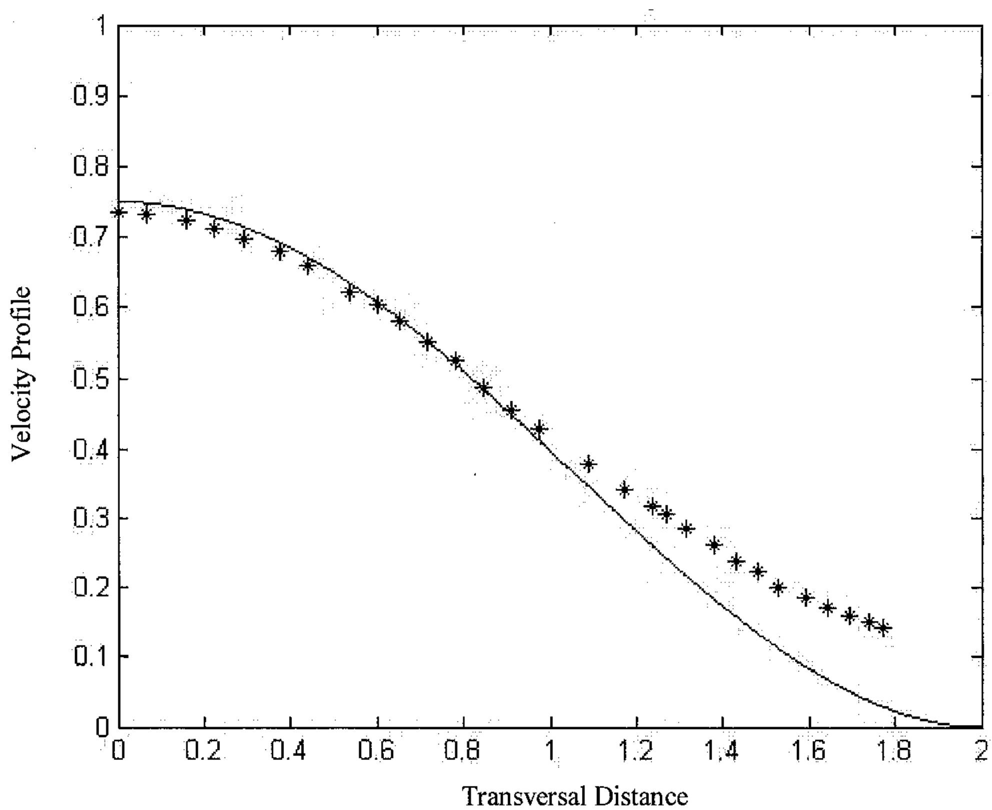


Figure 5.3: Velocity profile in transverse coordinate performed by analytical solution (solid line) and the experimental [106] data (points).

5.2 Turbulent Flow Cases

The third validation case is a water jet of diameter D = 10 cm and discharge velocity 10, 20 and 30 m/s [107]. Fig. 5.4 shows the measured radial profile of the normalised time-mean axial velocity at transversal locations for the turbulent round jet. It is seen that the streamwise velocity profile is similar to that of the laminar cases and can be well approximated by a half-Gaussian distribution.

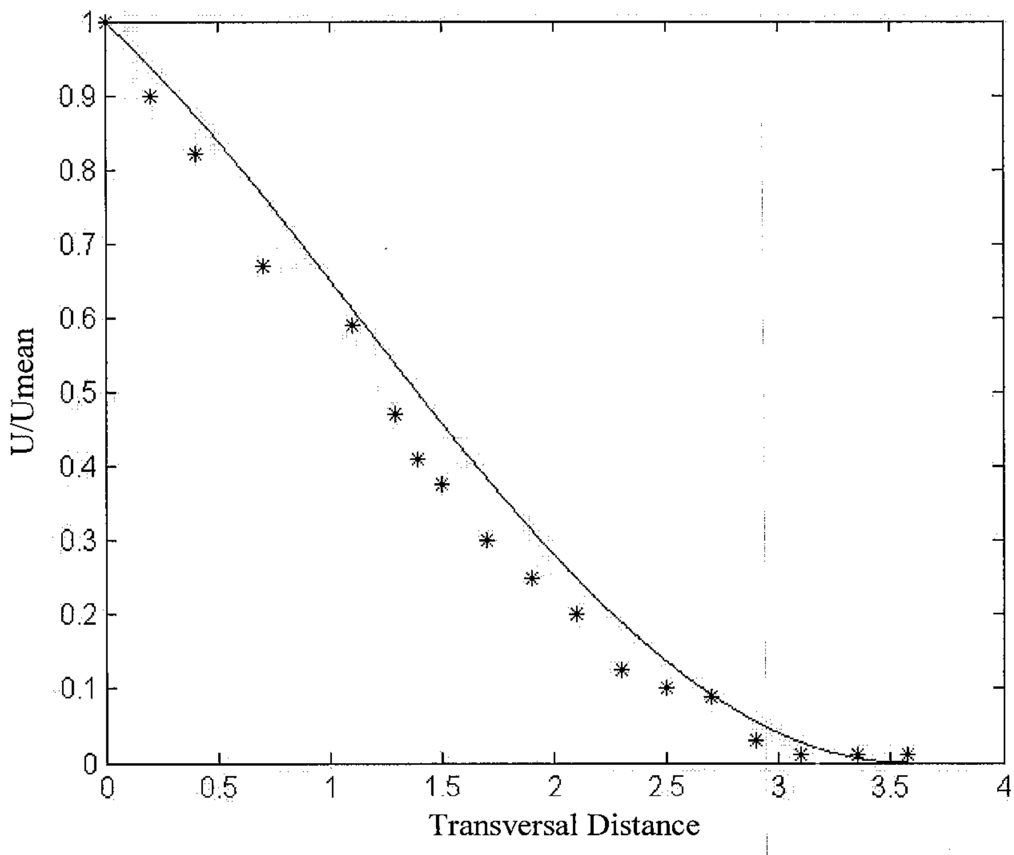


Figure 5.4: Turbulent velocity profile along transversal direction produced by analytical solution (solid line) and the experimental [107] data (points).

In Fig. 5.5, the calculated centerline velocity variation for the round turbulent jet is plotted against the measured values. The experimental results show clearly the existence of a potential core for about three diameters from the source, and the predicted variation confirms the experimental observation well. Previous experimental and analytical transverse velocity [108,109] shows similar trends to the case under discussion. In their work it could be seen that the transverse variation is similar, the data at different sections lie nicely onto one curve and can be well-approximated by the Gaussian distribution.

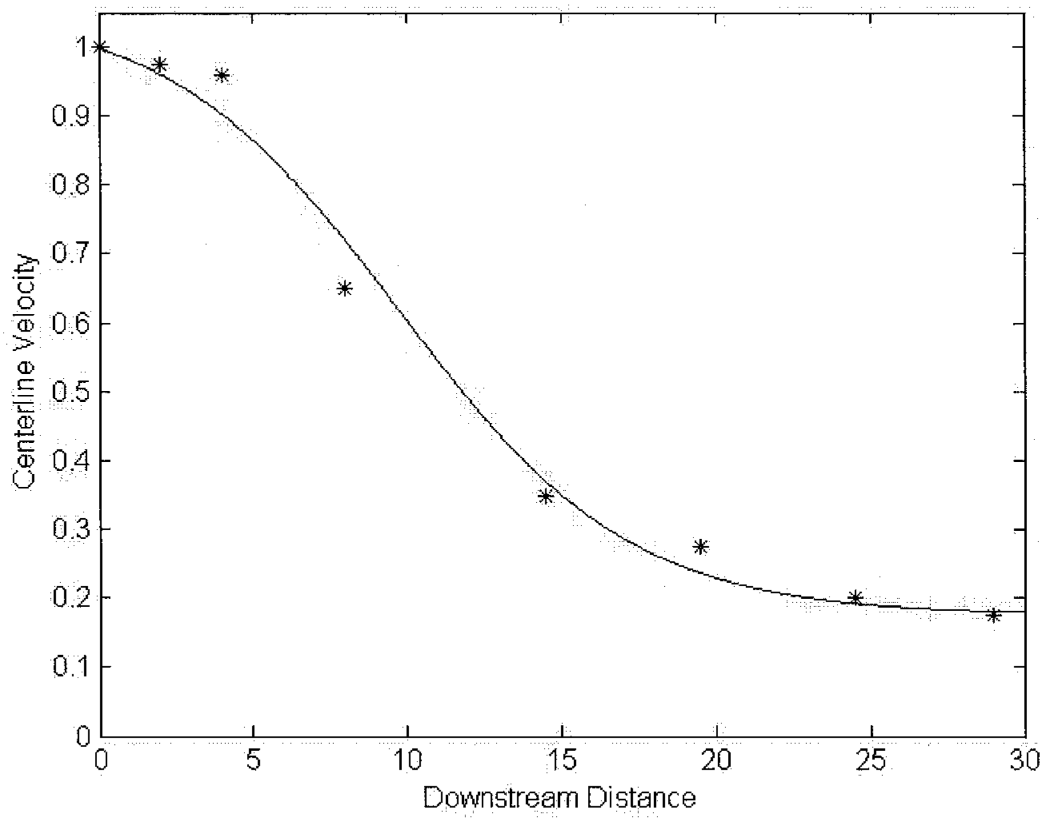


Figure 5.5: Turbulent decay velocity along downstream direction produced by analytical solution (solid line) and the experimental data (points) [107]

It is well known that turbulent flows are much more complicated than laminar flows, and thus some naïve prediction approaches will fail for turbulent flows even if they were successful for simple laminar flows. Therefore, the analytical solution needed to go through a second stage of validation against turbulent flow cases. The first turbulent flow case chosen for this validation stage is a boundary layer in atmospheric flow experiment of Farrel and Iyengar [110]. In their experiment, data were produced in a 1.7 m wide, 1.8 m high and 16 m long test section of the St. Anthony Falls Laboratory tunnel. The experimental technique was based on the use of quarter-elliptic, constant-wedge angle spires with height of 1.2 m and a castellated barrier wall to produce the necessary initial momentum defect in the boundary layer, followed by a fetch of roughness elements representative of the terrain under consideration. Fig. 5.6 gives a comparison of the calculated boundary layer velocity profile produced by the analytical solution and the measured profile from the experiment. The analytical results are in good agreement with the experimental data. Note that analytical solution described here is similar to the famous Blasius solution for boundary layer flows. Blasius solution for rectangular coordinate follows,

$$2f''' + ff'' = 0 \tag{5.5}$$

where all parameters above are non dimensional. Equation (5.5) is a class of quasi linear differential equation and similar to (4.13) for in its asymptotic limit and its solutions resemble previous solutions, thus it is not surprising that analytical solutions performed here can describe boundary layer flows.

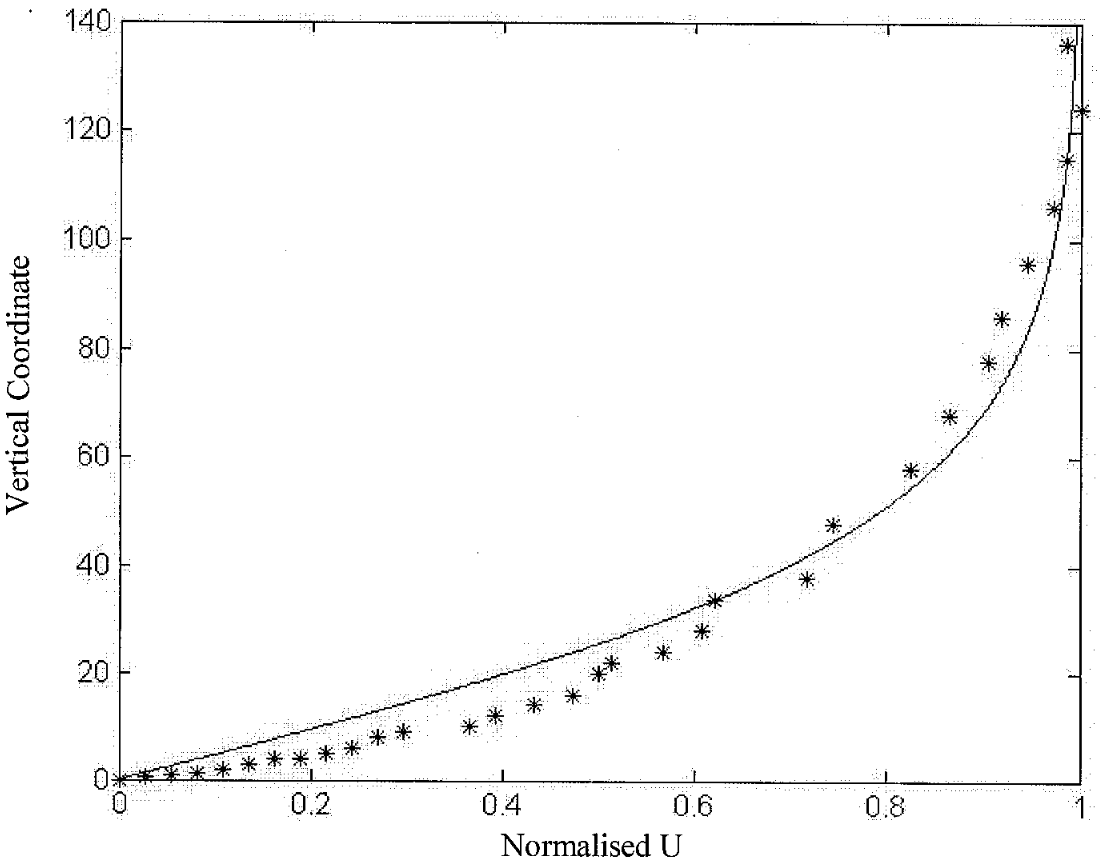


Figure 5.6. Trend of boundary layer velocity profile produced by analytical solution (solid line) and measured [110] values (points).

5.3 Combustion Case

The next challenging case used for validation in this stage is the recently published combustion experiment due to Cuoci et al. [111]. The fuel is fed in a central tube (3.2 mm internal diameter and 1.6 mm wall thickness), centered in a 15 cm x 15 cm square test section, 1m long, with flat Pyrex windows on the four sides. The fuel molar composition is 39.7% CO, 29.9 H_2, 29.7 N_2 and 0.70 CH_4. Ammonia was added in different amounts up to 1.64%; in the absence of ammonia, methane was not included in the fuel mixture. The average fuel flow velocity was 54.6 m/s with a resulting Reynolds number of ~8500; the inlet flow air velocity was 2.4 m/s. The inlet temperature of both streams is ~300K. Several radial profiles of velocity, temperature and species concentrations are available at different distances from the fuel inlet.

As shown in fig. 5.7, the analytical solution could reproduce the velocity change throughout the axial line with good agreement with the measured values. Detailed analysis for this case needs other equations (energy, species and thermodynamic state) to be solved simultaneously in order to describe turbulent-reaction interactions properly. This is of course a very challenging task and less tractable by considering that full mathematical theory for the Navier-Stokes equations is not yet complete. However, the comparison here is to show the potential of the simple analytical solution to tackle complex cases.

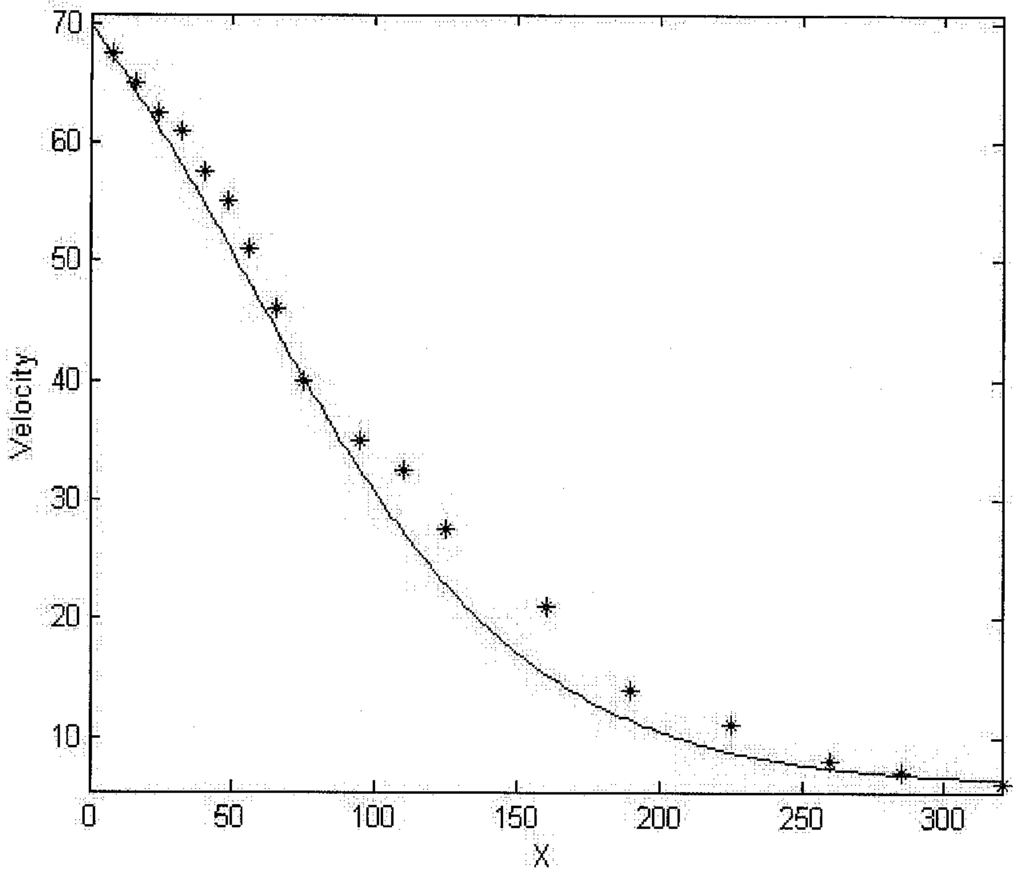


Figure 5.7: Measured mean axial velocity along flame centre line (points) [111] against the analytical solution (solid line)

5.4 Numerical Case

The next complex case is a comparison of the analytical solution against a numerical work in plane channel flow of Lammers et al. [112]. The numerical investigation presented here is using lattice boltzmann kinetic scheme, which discretise Boltzmann equation and then sum particles up to the hydrodynamics limits. Similar to the previous experimental cases, the analytical solution is found able to follow the non dimensional mean velocity profile. As shown in fig. 5.8, the deviation is obviously found in the generating zone which might be due to the raising of the reaction back the small scales to the more big ones. The result is also depicting the great potentiality of the analytical solution in tackling complex cases.

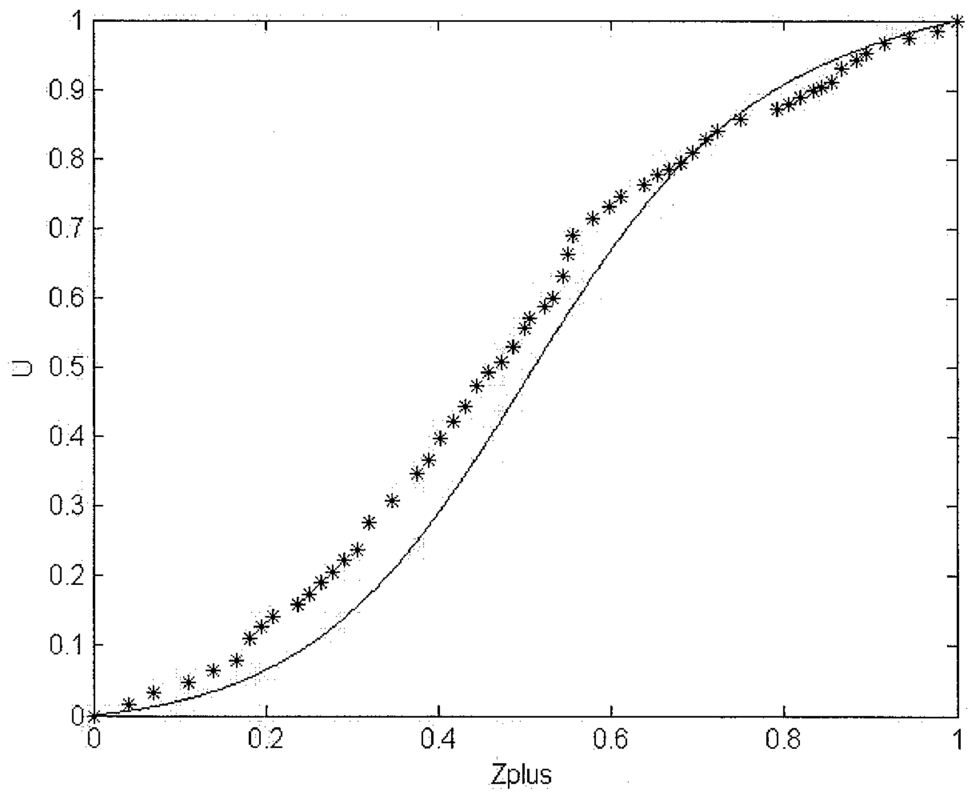


Figure 5.8: Numerical simulation of mean velocity profile (points) [112] against the analytical solution (solid line)

Chapter 6

Conclusions and Future Work

6.1 Main Conclusions

As a concluding remark, it is proved that the boundary value problems admit trivial solution only with special condition of zero rate of energy in the whole domain. The analyticity of the solutions can also be used to investigate the property of sharing regions which states that the solutions tend to be a constant value. A violation to this condition is observed to become a possible source of turbulence. Therefore, if turbulence is produced by generation of energy following the problem investigated here, it is reasonable to conclude that turbulent solutions may come from the boundary value problems of the Navier-Stokes equations as was stated previously in [92] as well.

It is concluded that the classes of solution $\vec{V} = \vec{\nabla}\times\Phi$ and $\vec{V} = \vec{\nabla}\Phi + \vec{\nabla}\times\Phi$ will transform the Navier-Stokes equations into the class of linear elliptic differential equations when analysis is conducted in vorticity form. An analysis of linear differential equations can be utilised to show that the solutions exist for at least $\Gamma \in L^2(\mathbb{R}^3)$ and $\Gamma \in H^1(\mathbb{R}^3)$ in $(0,T)$ which imply $\Phi \in H^3(\mathbb{R}^3)$ to ensure the regularity. Moreover, the uniqueness problem is also solved. Therefore, based on the condition discussed in [41], the class of the above solutions will also satisfy the Navier-Stokes equations. It is also important to mention here that the situation in $V = \nabla\Phi + \nabla\times\Phi$ is weaker than in $V = \nabla\times\Phi$ since it needs further assumption to become (3.13).

Analytical solutions of the three-dimensional incompressible Navier-Stokes equations are introduced in this thesis. First solution is derived using a four coordinate transformation without decomposition of potential function. The problem was reduced to the class of Riccati equation and had a well defined solution.

Second solution is derived using a potential function and a transformed coordinate in the form $\Phi = P(x,\xi)R(\xi)$. Since the explicit solution for R is obtained through the continuity equation, the potential function is then substituted into the Navier Stokes equations to reduce it to a class of nonlinear ordinary differential equations. Two different particular analytical solutions of P could be derived. The solutions for P for a zero pressure gradient case and for a constant pressure gradient case are found to be mathematically similar. The solution could be extended to a more general one based on the given particular solutions.

Third solution was obtained using similar procedure applied to a more general decomposition of potential function $\Phi = P(x,y,\xi)R(y)S(\xi)$ in the Navier-Stokes and vorticity equations. The explicit solutions for R and S are obtained through the continuity equation. The potential function is then substituted into the Navier-Stokes equations to reduce it to a class of nonlinear ordinary differential equation in term of P, where the pressure term is represented as a general functional form. General solution for P is derived based on the known particular solution by using the novel method for finding closed-form solutions of linear differential equations. The solution was regularised and proved to be unique.

Fourth solution was obtained using the vorticity equations where the decomposition of potential function $\Phi = P(x,y,\xi)R(y)S(\xi)$ was also implemented. The problem then reduced to the polynomial equations which were solved by the novel method. The pressure relation was solved by applying the velocity vector into the Navier-Stokes equations to complete the solutions. The solution was also proved to be unique.

As for the coordinate transformation, selection of variables in the potential function can be interchanged from the beginning. Instead of using the coordinate relation (2b) and potential function (4a), the following expression can be used,

$$\Phi = P(y,\xi)R(\xi), \quad \xi = lx + mz - \varsigma t, \quad \xi = lz + mx - \varsigma t \quad \text{or} \quad \Phi = P(z,\xi)R(\xi), \quad \xi = lx + my - \varsigma t, \quad \xi = ly + mx - \varsigma t \tag{6.1}$$

and

$$\Phi = P(\xi,y,z)R(y)S(z), \; \xi = kx - \varsigma(t) \; \text{or} \; \Phi = P(x,\xi,z)R(\xi)S(z), \; \xi = ky - \varsigma(t) \tag{6.2}$$

In particular, it is reasonable that other classes of nontrivial exact solutions may still be developed from the original Navier-Stokes equations by more complex procedures, to shed more light on the properties of the exact solutions [10]. The generality of $\varsigma(t)$ will make the obtained solutions open for further investigations.

The basic analytical framework for turbulent free jets, boundary layer, channel flow and combustion has also been presented. The governing equations based on the exact solutions to the incompressible Navier-Stokes equations are developed. It gives hindsight that turbulence closure can be achieved either by modeling and exact solutions. Although there are different characteristic properties of flows, the predictions are shown to be in excellent agreement with experimental and numerical data. In fact, based on this physical insight, most of the characteristic properties could have been deduced by a priori reasoning alone within general equations of fluid dynamics.

6.2 Future Work

The contributions of analytical methods to fluid dynamics research are impressive. Most of prior analysis and development are largely based on simplified equations such as two dimensional Navier-Stokes equations or linear advection and diffusion equations. The progress in nonlinear analysis has contributed a new trend in mathematics and physics. Unfortunately, this field is not famous research area and it should be brought together from diverse disciplines and with diverse viewpoints and new ideas can therefore be tested in a better way. For example, it is not clear whether exact solutions will alone be able to explain the formation of singularity in the Euler and Navier-Stokes equations. However, the future of research in deriving exact solutions of fluid dynamics equations appears to be promising. The greatest strength of exact solution is the simplicity of the function which then allows rapid calculations to the flow under studies. Exploiting this strength and using it to examine turbulent flows such as those discussed in chapter 5 appeared to be efficient in describing the flows. The diverse areas in the turbulent flows i.e. flow control, high-speed compressible flows, aeroelastic and reacting flows will likely see significant progress in the next future. The fast current progress in computer hardware will also give the additional stimulus to the need for exact solutions in order to test highly efficient numerical codes. Exact solutions can be used to improve statistical samples, and to consider a wide range of other physical parameters.

Hence, the future research of the exact solutions to the Navier-Stokes equations should be put in the following purpose,

(i) To regularise the Navier-Stokes equations, in the sense that the method should transform the Navier-Stokes equations from nonintegrable into integrable class and to show the singularity character which is possible in the equations, at least for incompressible cases.

(ii) To expand the analysis to the universal properties of turbulence known to experiments. Universal structure is connected to the small scale which contains the significant dynamics of turbulent flows. The analysis should interpret the Kolmogorov theory and related to the solutions of the Navier-Stokes equations.

(iii) To use the analytical solutions as a base to help evaluating the numerical codes. They should also be used to improve/explain the numerical schemes and relate them with other parameters such as perturbation and asymptotic analysis.

As the flow geometries become more complex, the analytical methods used in fluid dynamics research will have to evolve more. Engineers of computational fluid dynamics have much experience with complex geometries, and much can be learned about techniques from them. However, the significantly higher accuracy required by exact solutions must be kept in mind. Nonlinear methods of analysis and development are likely to prove very productive but still have to be developed more to reveal the secret of the Navier-Stokes equations.

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